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An Exploratory Study of Fifth-Grade Students' Reasoning About the Relationship Between Fractions and Decimals When Using Number Line-Based Virtual Manipulatives

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AN EXPLORATORY STUDY OF FIFTH-GRADE STUDENTS' REASONING ABOUT THE RELATIONSHIP BETWEEN FRACTIONS AND DECIMALS WHEN USING NUMBER LINE-BASED VIRTUAL MANIPULATIVES

by

Scott Smith

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ABSTRACT

An Exploratory Study of Fifth-Grade Students' Reasoning About the Relationship Between Fractions and Decimals When Using Number Line-Based Virtual Manipulatives

by

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Utah State University, 2017

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Understanding the relationship between fractions and decimals is an important step in developing an overall understanding of rational numbers. Research has demonstrated the feasibility of technology in the form of virtual manipulatives for facilitating students' meaningful understanding of rational number concepts. This exploratory dissertation study was conducted for the two closely related purposes: first, to investigate a sample of fifth-grade students' reasoning regarding the relationship between fractions and decimals for fractions with terminating decimal representations while using virtual manipulative incorporating parallel number lines; second, to investigate the affordances of the virtual manipulatives for supporting the students' reasoning about the decimal-fraction relationship.

The study employed qualitative methods in which the researcher collected and analyzed data from fifth-grade students' verbal explanations, hand gestures, and mouse cursor motions. During the course of the study, four fifth-grade students participated in an initial clinical interview, five task-based clinical interviews while using the number line-based virtual manipulatives, and a final clinical interview. The researcher coded the data into categories that indicated the students' synthetic models, their strategies for converting between fractions and decimals, and evidence of students' accessing the affordances of the virtual manipulatives (e.g., students' hand gestures, mouse cursor motions, and verbal explanations).

The study yielded results regarding the students' conceptions of the decimal-fraction relationship. The students' synthetic models primarily showed their recognition of the relationship between the unit fraction 1/8 and its decimal 0.125. Additionally, the students used a diversity of strategies for converting between fractions and decimals. Moreover, results indicate that the pattern of strategies students used for conversions of decimals to fractions was different from the pattern of strategies students used for conversions of fractions to decimals. The study also yielded results for the affordances of the virtual manipulatives for supporting the students' reasoning regarding the decimal-fraction relationship. The analysis of students' hand gestures, mouse cursor motions, and verbal explanations revealed the affordances of alignment and partition of the virtual manipulatives for supporting the students' reasoning about the decimal-fraction relationship. Additionally, the results indicate that the students drew on the affordances of

alignment and partition more frequently during decimal to fraction conversions than during fraction to decimal conversions.

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PUBLIC ABSTRACT

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CHAPTER I

INTRODUCTION

Fractions and decimals are each important ways of symbolically representing rational numbers; furthermore, fractions and decimals are each fundamental subjects in the mathematics curriculum that students should learn during grades three through eight. According to the Common Core State Standards for Mathematics (CCSSM), by the eighth grade students should have a strong understanding of the relationship between fractions and decimals (Common Core State Standards Initiative, 2010). The CCSSM specify that students should begin understanding the relationship between fractions and decimals as soon as they begin learning about decimals in the fourth grade. In addition, students' understanding of the relationship between fractions and decimals can substantially contribute to their *rational number sense*, considered important by a number of mathematics educators for students' reasoning with rational numbers (Lamon, 2007; Sowder, 1995).

In spite of the importance of developing an understanding of the relationship between these two ways of symbolizing rational numbers emphasized in curriculum standards, data from the National Assessment of Educational Progress (NAEP) indicate that many students have only a superficial understanding of the decimal-fraction relationship. For example, NAEP data from 2004 indicate that only 42 percent of twelfth-grade students were able to correctly convert the repeating decimal 0.3333… to a fraction, and only 35 percent of twelfth-grade students were able to convert the decimal 0.029 to a fraction (Rutledge, Kloosterman, & Kenney, 2009). These results indicate that students are completing their K-12 education with an inadequate understanding of the decimal-fraction relationship.

A considerable body of research has investigated how students learn fraction and decimal concepts from different types of representations. Researchers recognize that number line representations have affordances that make them useful for facilitating students' understanding of fractions and decimals, such as depicting the order and density properties of fractions and decimals (Siegler et al., 2010). Number lines are potentially effective to facilitate students' learning of the decimal-fraction relationship, since parallel number lines can depict this relationship (Fosnot & Dolk, 2002; National Council of Teachers of Mathematics, 2000). However, research has not explored the full range of possibilities for using number line representations for teaching students rational number concepts, such as the decimal-fraction relationship.

Statement of Purpose

There are two purposes for conducting this dissertation study. The first purpose is to investigate a sample of fifth-grade students' reasoning regarding the relationship between fractions and decimals for fractions with terminating decimal representations while using virtual manipulative incorporating parallel number lines. The second purpose is to investigate the affordances of the virtual manipulatives for supporting the students' reasoning about the decimal-fraction relationship, using the categories of affordances identified by Moyer-Packenham and Westenskow (2016) specifically for virtual manipulatives.

Research Questions

The first research question is an overarching question with two sub-questions that concerns students' conceptions of the decimal-fraction relationship. The second research question concerns the affordances of the number line-based virtual manipulatives for supporting students' reasoning about the decimal-fraction relationship.

1. What are fifth-grade students' conceptions of fractions as decimals and decimals as fractions for fractions with terminating decimal representations?

1a. What synthetic models do students construct regarding the relationship between fractions and decimals, while working on tasks involving number line-based virtual manipulatives?

1b. What is the evidence of students' reasoning about the relationship between fractions and decimals for fractions with terminating decimal representations?

2. What are the affordances of number line-based virtual manipulatives for supporting students' reasoning about the relationship between fractions and decimals as indicated by their hand gestures, mouse cursor motions, and explanations?

Definition of Terms

For the purposes of clarity, the following terms are used in this study.

Burlamaqui and Dong (2014) define *affordances* as “cues of the potential uses of an artefact by an agent in a given environment”. In this study, the categories of affordances identified by Moyer-Packenham and Westenskow (2016), specifically for virtual manipulatives, are how the affordances are defined.

An *external representation* of a mathematical concept is an embodiment of the concept that retains salient features of the mathematical concept, and provides a visual model of the mathematical concept (Goldin, 2002).

Vosniadou, Vamvakoussi, and Skopeliti (2008) define children’s *framework theories* as the theories children develop from infancy, which form a coherent explanatory system. Developmental psychologists have established that children form at least four distinct framework theories regarding language, mathematics, physics and psychology.

Vosniadou (1994) defines *initial models* as students’ initial conceptions of concepts or scientific phenomenon before instruction that are based on everyday experience.

According to Lamon (2007), *rational number sense* is characterized as: intuitive understanding of the relative sizes of rational numbers; qualitative and multiplicative thinking about rational numbers; the ability to move flexibly between interpretations and representations of rational numbers; and the ability to solve proportions involving rational numbers.

An *internal representation of knowledge* refers to the cognitive structure of knowledge in the human mind (Hiebert & Carpenter, 1992).

Vosniadou, Vamvakoussi, and Skopeliti (2008) regard a *mathematical model of rational number* to be a mathematically accurate understanding of rational number properties.

A *synthetic model* is a conception resulting from the enrichment of prior knowledge through the additive learning of new information that is incompatible with the prior knowledge, which results in an inaccurate mental model (Vosniadou, 1994).

Moyer-Packenham and Bolyard (2016) define a *virtual manipulative* as "an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" (p. 5).

CHAPTER II

LITERATURE REVIEW

The Decimal-Fraction Relationship

A rich understanding of rational numbers as quantities is an important learning goal of the middle-grades curriculum (Behr, Harel, Post, & Lesh, 1992; Carpenter, Fennema, & Romberg, 1993; Hiebert & Behr, 1988; Lamon, 2007; Sowder, 1995), where students need to understand that every rational number can be represented symbolically in several equivalent ways (Kilpatrick, Swafford, & Findell, 2001). The Common Core State Standards for Mathematics (CCSSM) specify students' should learn about the decimal-fraction relationship as early as fourth grade (Common Core State Standards Initiative, 2010). NCTM's *Principles and Standards for School Mathematics* recommends that in grades 3-5 students should be able to "recognize equivalent representations of the same number" (p. 148) and to "recognize and generate equivalent forms of commonly used fractions, decimals, and percents" (National Council of Teachers of Mathematics, 2000, p. 148). Siegler et al. (2010) recommend that by the eighth grade students should understand that rational numbers can be represented as fractions, decimals, and percentages, and be able to translate rational numbers into these different symbolic forms. Students' understanding of the relationship between decimals and fractions contributes substantially to their rational number sense (Sowder, 1995), seen as supporting students' ability to reason proportionally (Lamon, 2007), which is necessary for learning algebra (Kaput & West, 1994).

Despite its importance, assessment results indicate that students struggle to develop an understanding of the relationship between fractions and decimals. For example, Kloosterman (2010) reported that only 40 percent of the twelfth-grade students who took the 2004 LTT NAEP were able to convert the repeating decimal 0.333333… to a fraction, and only 29% of the twelfth-grade students were able to convert 0.029 to a fraction. Hiebert and Wearne (1986) observe 25% of fifth-grade students held a misconception that the decimal 0.09 converts to the fraction 0/9. Markovits and Sowder (1991) observed many students did not believe it was possible to compute the sum ½ + 0.5, reasoning that ½ and 0.5 are different types of numbers and cannot be combined. Furthermore, many of these students could not determine whether 1.7 and 1/7 were the same or different, and similarly for the numbers 0.5 and 6/12.

Before students can understand the relationship between fractions and decimals, they must first understand that both fractions and decimals represent quantities (Kilaptrick et al., 2001). However, studies indicate students encounter difficulties understanding fractions and decimals as numeric magnitudes (Hiebert, 1992; Lamon, 2007). One monumental barrier in students' understanding of fractions as quantities is their *whole number bias* (Behr, Wachsmuth, Post, & Lesh, 1984; Ni & Zhou, 2005; Post et al., 1985; Siegler & Pyke, 2013). As students learn about whole number operations in the early grades, they develop misconceptions concerning numeric density and the need for and nature of rational numbers. For instance, students who think numbers can only be used to quantify discrete quantities will argue that 0.45 is less than 0.412 because 45 is less than 412 (Hiebert & Wearne, 1986); and 3/5 is less than 3/8 because 5 is less than 8

(Post, Cramer, Behr, Lesh, & Harel, 1993). Moreover, whole number biases can lead students to think the fraction following 2/5 is 3/5, and that the decimal following 0.32 is 0.33, reflecting little understanding of the density of rational numbers (Vamvakoussi & Vosniadou, 2010).

The above named whole number biases contribute to students' difficulties with equivalence relations between fractions and decimals, (e.g., 2/3 = 4/6, or 0.25 = 25/100 = ¼, Hiebert & Wearne, 1986). Such misconceptions are often deeply entrenched, persistent, and resistant to change (Vamvakoussi & Vosniadou, 2010). Yet, whole number-based misconceptions about either fractions or decimals need to be resolved before students can develop a mathematically accurate understanding of the decimal-fraction relationship. Flawed conceptions of either decimals or fractions will carry over into conceptions of the relationship between decimals and fractions. Put differently, for students to successfully understand that different names, notations, or representations can be used to refer to the same quantity, they must change their conception of what numbers are by expanding their conception of the nature of numbers, incorporating key properties of fractions and decimals in their understanding (Steffe & Olive, 2010, Siegler, Fazio, Bailey, & Zhou, 2013; Tzur, 2007).

Conceptual Change Theory and Student Conceptions

Science education researchers have previously used conceptual change approaches to investigate students' conceptions of scientific phenomena (Vosniadou, Vamvakoussi, and Skopeliti, 2008). Several researchers in science education observed students' overcoming of misconceptions often paralleled major breakthroughs in the history of science. These researchers theorized that students might overcome their science-based misconceptions, in a manner analogous to conceptual revolutions that have occurred in the fields of science, through processes of conceptual change. After researchers successfully applied conceptual change theories to study students' learning in numerous science domains, they began applying conceptual change theories to students' understanding and learning of mathematical subjects where students frequently have common and persistent misconceptions, such as rational numbers (Vamvakoussi & Vosniaou, 2010). In particular, Vosniadou and colleagues applied a conceptual change framework in several investigations of the misconceptions students commonly develop while learning fractions, decimals, and the decimal-fraction relationship (Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosnidou, 2010).

The conceptual change framework has been successfully applied by researchers to interpret and explain students' misconceptions when learning scientific and mathematical topics, and to investigate the role of prior knowledge in the formation of misconceptions (Vosniadou, Vamvakoussi, & Skopeliti, 2008). Conceptual change researchers contrast learning via conceptual change processes with learning through enrichment processes, where enrichment learning is viewed as an additive process of new information being added onto students' knowledge without any conceptual restructuring. When students enrich their deeply entrenched prior knowledge with new knowledge that conflicts with their prior knowledge, the results are misconceptions (Vamvakoussi & Vosniaou, 2010; Smith, Solomon, & Carey, 2005). During the initial learning of rational numbers, researchers have found that prior knowledge of whole numbers interferes with the learning of fractions, decimals, and the decimal-fraction relationship (Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2010).

Vosniadou (1994) observes that developmental research demonstrates that as children grow from infancy, they form very well defined intuitive theories about the world around them, which Vosniadou refers to as *framework theories*. According to Vosniadou, Vamvakoussi, and Skopeliti (2008), there are at least four framework theories children form about the world around them, including frameworks for physics, psychology, language, and mathematics. These *framework theories*, by being extremely consistent, allow children to make predictions about the world they observe around them, and by the time children reach school age these *framework theories* have become deeply entrenched. Vosniadou (1994) refers to these initial conceptions as *initial models*. As children learn about a new subject domain through enrichment processes by adding information onto their existing framework theory, the results are often misconceptions Vosniadou (1994) refers to as *synthetic models*. According to Vosniadou, a synthetic model is a conception created by a student as they attempt to link their initial perspective to the scientifically or mathematically correct perspective not yet fully understood by the student.

By the time children have reached school age, they have formed a framework theory of numbers as counting numbers (Vamvakoussi & Vosniadou, 2010). Because of such framework theories of numbers, children come to hold several very specific beliefs about number properties, such as numbers are used only for counting discrete objects, every number has a successor, and every number has only one symbolic representation. As students initially learn about fractions and decimals, they enrich their framework theory of numbers as counting numbers, resulting in numerous misconceptions, such as the ones described earlier.

As opposed to pure enrichment, students need to undergo a conceptual change to restructure and expand their understanding of what numbers are and what they are used for, which is necessary for a correct conceptualization of both fractions and decimals. Students are able to successfully accomplish this change through a gradual replacement of the beliefs held in their original framework, resulting in the formation of correct conceptions of fractions and decimals, and consequently achieving the desired conceptual change. Only by making such a conceptual change will students be able to fully understand appropriate properties of fractions and decimals, such as the density property, that fractions and decimals do not have successors, and, ultimately, that decimals and fractions can serve as notions or representations for the same quantity. Following Vosniadou, Vamvakoussi and Skopeliti (2008), in this study the researcher refers to a mathematically accurate understanding of rational number as a *mathematical model of rational number*.

Conceptual Stepping Stones for the Decimal-Fraction Relationship

Students develop meaningful understanding of fraction and decimal concepts, and the resulting understanding of the decimal-fraction relationship, on a foundation of several conceptual stepping-stones. Research indicates five conceptual stepping-stones, or understandings, students must construct: a) notions of the unit or whole for both fractions and decimals; b) notions of *unit* fractions and decimals quantities; c) notions of *non unit* and *benchmark* fractions and decimals quantities; d) partitioning; e) and iterating unit numbers to create non-unit quantities. An integrated understanding of each of the above conceptual stepping-stones is a necessary condition for understanding how fractions and decimals can equivalently represent the same quantity. This section describes the importance of these conceptual stepping-stones for understanding fractions and decimals, by explaining how these conceptual stepping-stones support students' understanding of the decimal-fraction relationship.

Understanding the Unit or Whole

For fractions and decimals, the *unit* or *whole* refers to the number one (Lamon, 2007), which represents the whole of the relevant quantity. Being able to identify the unit or whole is an essential part of understanding that fractions and decimals represent quantities, since the value of all other fractions and decimals are determined relative to the value of the unit or whole. For instance, the unit fraction 1/n results from segmenting or partitioning the unit or whole into *n* equal parts, so that the value of unit fractions are determined relative to the unit or whole. Understanding the value of a fraction such as ¾ requires understanding that ¾ is a composite fraction created through an iteration of ¼, which entails a coordination of the value of ¼ and ¾ relative to the unit or whole. Students' rational number sense is supported by increasingly sophisticated methods of composing and recomposing the unit or whole from subunits (Lamon, 1994; Lamon, 1996; Lamon, 2007; Steffe & Olive, 2010). Lamon (1996) refers to composing and recomposing the unit or whole in terms of subunits as *unitizing*, and found that this skill is an important part of multiplicative reasoning.

Magnitude knowledge of the unit or whole is a necessary part of multiplicative reasoning with fractions and decimals. When a student does not understand the value of a fraction or decimal in relation to the *unit*, the student can attain at best an additive understanding of the value of the number. For instance, if a student does not understand the value of 4/5 in relation to the unit or whole, then the student is only able to focus on the number of parts in the fraction, namely the numerator 4, which is an additive understanding of the value of the fraction (Mack, 1993).

Understanding Unit Fraction and Decimal Magnitudes

Research indicates unit fractions of the form *1/n* play a key role in facilitating students' understanding that fractions represent quantities or magnitudes (Norton & McCloskey, 2008; Steffe & Olive, 2010). According to Siegler, Fazio, Bailey and Zhou (2013), unit fractions play a significant role in the development of students' understanding of fractions as magnitudes. Furthermore, the researchers observe the prominent role unit fractions play in students' learning of proper fractions (fractions whose value is less than 1), and how an understanding proper fractions provides a strong foundation for students' learning of other types of fractions, such as improper fractions (fractions whose value is greater than 1) and mixed numbers (numbers such as 1¾). Indeed, Siegler et al. argue that proper fractions play an influential role in students' overall fraction learning, where students need a strong understanding of proper fractions before being able to develop understanding of improper fractions. According to Norton and McCloskey (2008), students can gain an understanding of fractions as representing quantities or magnitudes by understanding that every fraction is an iteration, and thus a multiple, of a unit fraction. Research has yet to establish the role that unit decimals may play in students' understanding of the magnitude or quantity represented by the decimal equivalents of non-unit fractions.

Understanding Benchmark and Non Unit Fractions and Decimals

Benchmark values of rational numbers play a fundamental role in the development of students' meaningful understanding of fractions and decimals. Benchmark values of fractions are simple and commonly used fractions, such as ½, 1, ¼, and ¾, and their respective equivalent decimal values of 0.5, 1, 0.25, and 0.75 for decimals (Sowder, 1995). Research by the Rational Number Project indicates that benchmark values of fractions play a key role in facilitating students' understanding of fractions as quantities or magnitudes. In particular, the Rational Number Project found that students often use benchmark values as a strategy when attempting to order fractions, such as estimating whether a fraction is greater than or less than ½ (Behr et al., 1984; Behr, Wachsmuth, & Post, 1985; Cramer, Post, & delMas, 2002). Additionally, Smith (1995) found students' use of benchmarks for estimation purposes was a characteristic of expertise in reasoning with rational numbers, where students with expertise spontaneously use estimation strategies involving benchmarks.

Researchers have investigated the effectiveness of experimental instruction incorporating benchmarks numbers of fractions and decimals. In the study by Sowder and Markovits (1989), the researchers engaged students in instruction emphasizing strategies for using benchmarks to make estimations with rational numbers. The researchers found the instruction resulted in the improvement of students' magnitude knowledge of rational numbers. In the study by Moss and Case (1999), the researchers measured the effectiveness of an experimental curriculum designed to teach fourth-grade students about fractions, decimals, and percentages, so that the students would have an understanding of the relationship between these three symbolic systems of rational numbers. Benchmark numbers played a prominent role in the Moss and Case study, as their instructional approach involved the facilitation of students' understanding of the relationship between fractions, decimals, and percentages for benchmark numbers.

Equivalence and the Decimal-Fraction Relationship

An understanding of fractions as representing magnitudes or quantities is essential for students to understand fraction equivalence. Indeed, a student will not be able to grasp the fact that two fractions are equal numbers if the student does not understand that the two fractions represent the same quantities. For instance, if a student has the misconception of a fraction as consisting of two whole numbers, such a student would not be able to understand the concept that two fractions can be equal.

It is essential for students to understand equivalent fractions before completely understanding the decimal-fraction relationship. For instance, consider the following decimal-fraction relationship: ¼ = 25/100 = 0.25. Before a student can understand the relationship between the fraction ¼ and the decimal 0.25, the student must understand that ¼ = 25/100, a relationship involving equivalent fractions. According to Kamii and Clark (1995), students' understanding of equivalent fractions is based on their ability to reason multiplicatively. For example, the multiplicative reasoning involved in understanding the equivalent fractions relationship ¼ = 25/100 is the student must understand that the relationship between 1 and 4 is the same as the relationship between 25 and 100. This relationship is one of multiplication, namely that 4 is 4 times greater than 1, and that 100 is 4 times greater than 25. Speaking more generally, we can say a student understands a fraction a/b is equivalent to ¼, if the student realizes the denominator b is 4 times greater than the numerator a, an example of multiplicative reasoning.

Partitioning: Supporting Notions of Unit and Unit Fraction/Decimal

Pothier and Sawada (1983) refer to partitioning as the process of dividing the unit or whole into parts. Pothier and Sawada investigated the partitioning strategies of students in grades K-3, and found there were four levels of understanding of the partitioning process: sharing, algorithmic halving, evenness (dividing the unit into an even number of pieces), and oddness (dividing the unit or whole into an odd number of pieces).

As children increase in the sophistication of their partitioning strategies, this supports a necessary idea about fractions and decimals, namely the necessity of creating equal-sized pieces when partitioning. Understanding that fractions are composed of an iteration of equal-sized unit fractions is an essential part of understanding fractions and decimals (Behr, Lesh, Post, & Silver, 1983). Furthermore, the meaningful learning of fraction and decimal concepts depends on an integration of counting and partitioning (Carpenter et al., 1993). Moreover, experience with partitioning supports students' understanding of the inverse relationship between the number of pieces in the partition and the size of the related unit fraction (Behr et al., 1992; Tzur, 2007), an important part of achieving an understanding that fractions represent quantities or magnitudes. In addition, Empson (1999) found first-graders' partitioning and sharing activities supported the development of basic ideas about fraction equivalence. According to Steffe (2003), as students increase in their sophistication of composing and recomposing the *unit* or *whole*, this supports their understanding of equivalent fractions.

Partitioning the unit or whole is an activity likely to support students' understanding of the relationship between fractions and decimals. As students create simultaneous partitions of the unit or whole using equivalent fractions and decimals, such partitions can support their understanding that fractions and decimals are each closely related ways of representing rational numbers.

Iteration: Supporting Notions of Non-Unit

Students understanding of how to iterate unit fractions of the form *1/n m* times to create a fraction *m/n* also supports students' rational number sense. The size or magnitude of the fraction *m/n* is determined by the number of iterations *m* of the unit fraction *1/n* (Norton & McCloskey, 2008). In addition, according to Steffe and Olive (2010), the process of creating fractions *m/n* by iteration of a unit fraction can facilitate the development of students' fraction language. Furthermore, Keijzer and Terwel (2001) conducted a case study of one student's learning of fraction concepts from number lines. The researchers observed that an early strategy the student invented was to generate fractions by iterating unit fractions on a number line.

Halving and Doubling

A strategy Moss and Case (1999) found to be highly effective in developing fourth-grade students' understanding of the relationships among fractions, decimals, and percentages was halving and doubling. For instance, as students worked with a variety of authentic, real-world materials, they constructed strategies involving halving and doubling that allowed them to find the relationship between different fractions and decimals. This is exemplified in the following type of reasoning: starting with the decimal-fraction relationship ½ = 0.5, by repeated halving a student is able to understand that ¼ = 0.25, and 1/8 = 0.125.

Disembedding

Another fundamental operation essential to understanding fractions as quantities is *disembedding*. Steffe and Olive (2010) identify *disembedding* as the mental activity of removing a part from a whole while keeping the whole intact, where the part and the whole are conceived of as separate entities. Steffe and Olive consider *disembedding* necessary for understanding part-whole comparisons. This occurs, for example, when a student realizes that 4/5 is greater than 3/4, because 4/5 is missing 1/5 from the whole, whereas 3/4 is missing 1/4 from the whole, and 1/5 is a smaller piece than 1/4.

What Has Been Done: Previous Research on the Decimal-fraction Relationship

The conceptual stepping-stones discussed above, along with tasks and instructional interventions, may work to support students' understanding of the decimal-fraction relationship. Yet, very few studies exist that have investigated elementary students' learning of and reasoning regarding the relationship between fractions and decimals. One such study is Moss and Case (1999).

Moss and Case were successful in teaching fourth-grade students to understand the relationship between rational numbers expressed as fractions, decimals, and percentages. The instructional approach of Moss and Case put substantial emphasis on benchmark numbers. Moreover, the students were able to develop a strategy of halving and doubling to further their understanding of the relationship between fractions, decimals, and percentages. However, Moss and Case did not investigate or document students' intermediate knowledge states or synthetic models as they came to understand the relationship between the different symbolic representations of rational numbers. In addition, Moss and Case did not consider the role unit fractions and decimals could play as students learn about the relationship between fractions, decimals, and percentages.

Vosniadou and colleagues used a conceptual change approach to extensively document that students' frequently develop different conceptions of fractions and decimals (Stafylidou & Vosniadou, 2004; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011; Vamvakoussi & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2007; Vamvakoussi & Vosniadou, 2010; Vamvakoussi, Vosniadou, & Van Dooren, 2013). The majority of the studies of Vosniadou and colleagues concern students' synthetic model and conceptual change as they come to understand the density property of both fractions and decimals (Vamvakoussi, Christou, Mertens, & Van Dooren, 2011; Vamvakoussi & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2007; Vamvakoussi & Vosniadou, 2010; Vamvakoussi, Vosniadou, & Van Dooren, 2013). Their research results indicate students frequently have different conceptions of the density property for fractions and decimals, and consequently have qualitatively different understandings of fractions and decimals. However, Vosnidou and colleagues did not investigate why many students have qualitatively different understandings of fractions, by not investigating causes of students' different understandings of fractions and decimals. If researchers understood better the reasons why students have different conceptions of fractions and decimals, we could better understand the reasons for students' difficulties in understanding the decimal-fraction relationship.

Internal and External Representations of Knowledge

The researcher draws on a framework of Hiebert and Carpenter (1992) to describe the structural aspects of mathematical knowledge, learning processes and how learning from representations occurs. The framework of Hiebert and Carpenter draws on three key assumptions from research in the cognitive sciences. The first assumption is "knowledge is represented internally, and that these internal representations are structured" (p. 66). One way Hiebert and Carpenter characterize students' *internal knowledge representations* is metaphorically as a network, where the nodes of the network are pieces of represented information, and the connections in the network represent relationships between the information.

Hiebert and Carpenter's second assumption is learning results in the connection of internal representations of knowledge in ways beneficial for understanding. Indeed, according to Hiebert and Carpenter, a mathematical concept is understood if its internal representation is part of the internal network of knowledge. The authors maintain that the greater the numbers of connections in an internal network of knowledge and the stronger the connections within the network, the greater the degree of understanding. Thus, we can characterize students' internal representations of knowledge of mathematical concepts as structured and organized networks of knowledge.

Hiebert and Carpenter make a distinction between internal and external representations, where internal knowledge representations are the cognitive structures of knowledge in a learner's mind, and external representations often assume the forms of spoken language, pictures, written symbols, and manipulative models (Lesh, Post, & Behr, 1987). Hiebert and Carpenter's third assumption is external representations can influence students' internal representations of knowledge, where external representations of mathematical concepts can facilitate and support students' learning of those mathematical concepts.

Mathematics educators have long perceived external representations as an effective means of making mathematical ideas understandable for students (Hiebert & Carpenter, 1992). According to Hiebert and Carpenter (1992), the instructional use of external representations of mathematical concepts can facilitate students' construction of mental models. Additionally, Goldin (2000, 2003) maintains that external representations should play a fundamental role in empirical investigations of students' reasoning and understanding of mathematical concepts, such as during task-based clinical interviews.

Part-Whole and Number Line Representations of Fractions and Decimals

The two most commonly used types of representations of fractions and decimals in the U.S. curriculum are part-whole representations and number line representations (Lamon, 2001; Fuchs et al., 2013). Part-whole representations can facilitate students' initial understanding of fractions by building on their informal knowledge gained from personal experiences, such as sharing (Mack, 1993; Sowder, 1995). However, some researchers have observed that there has been an overreliance on part-whole representations of fractions in the U.S. curriculum (Siegler et al., 2010; Sowder, Bezuk, & Sowder, 1993). Additionally, there are a number of weaknesses of the part-whole conception of fractions, in terms of the types of ideas reinforced by this representation. Mack (1993) points out that because of the discrete nature of part-whole representations students have a tendency to focus on the parts as discrete objects, not taking into consideration the multiplicative relationship between the numerator and denominator, resulting in students not attaining an understanding of fractions as quantities (Behr, et al., 1984). In addition, Kerslake (1986) argued that part-whole representations cannot be used to teach the ratio conception of fractions. Furthermore, researchers observed the difficulties students encounter understanding improper fractions when reasoning about fractions as parts of a whole (Mack, 1993; Thompson & Saldahna, 2003). Lamon (2001) maintains part-whole representations are not a sufficient foundation on which to construct an understanding of fractions and decimals, and other researchers have found overreliance on the part-whole representations can inhibit a complete understanding of fractions and decimals in the long run (Mamede, Nunes, & Bryant, 2005).

There are a number of benefits of using number line representations to teach concepts of fractions and decimals (Siegler, Thompson, and Schneider, 2011). Because of the geometrical nature of number lines, this type of representation captures the most salient properties of fractions and decimals, including: fractions and decimals do not have successors; the density property of fractions and decimals; fractions and decimals can be used to represent continuous quantities; and equivalence concepts of fractions and decimals (NMAP, 2008). Rittle-Johnson, Siegler, and Alibali (2001) found students were able to effectively learn decimal concepts and procedures from number lines, and number lines promoted students' knowledge of decimals as magnitudes. Number lines naturally lend themselves to the illustration of the addition and subtraction of rational numbers (Lamon, 2007), and can facilitate students' realization that rational numbers are quantities or magnitudes, which has been shown to substantially enrich students' conceptual understanding of fractions and decimals (Siegler, Thompson, & Schneider, 2011). Furthermore, number lines are representations that can be used to represent the different forms of rational numbers, including fractions, decimals, percentages, as well as whole numbers and real numbers, and can be used to depict the relationship between these different number systems (Siegler, Fazio, Bailey, & Zhou, 2013), making number lines valuable for facilitating students' understanding of the decimal-fraction relationship.

Furthermore, policy documents, including the NMAP Report (2008), a Fractions Guide published by the What Works Clearinghouse (Siegler et al., 2010), and the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), recommend increasing the use of number lines for fraction and decimal instruction. In particular, Siegler et al. (2010) emphasize how number lines can help students understand that fractions represent numbers with magnitudes, order and equivalence concepts of fractions, and facilitate students' understanding of the relationship between fractions and decimals. Additionally, the Common Core State Standards for Mathematics for the third grade recommend using number line representations of fractional quantities to facilitate students' understanding of fractions as magnitudes (Common Core State Standards Initiative, 2010).

Vamvakoussi and Vosniadou (2010) observe that instruction based on number lines can facilitate students' development of a mathematically accurate understanding of the relationship between fractions and decimals, since number lines can help students to develop a correct conceptual understanding of both fractions and decimals. Such representations are effective because of the human cognitive system's ability to create internal representations of knowledge embodying features of the represented concept. The human mind is able to manipulate mental representations to understand important properties of the represented concepts (Greeno, 1983). Because number lines are useful for representing both fractions and decimals, they are useful for teaching the relationship between these two number systems, as well as for investigating students' understanding of the decimal-fraction relationship. Moreover, number lines effectively model important properties of rational numbers, including the density property, the lack of successors, multiple symbolic representations, and the representation of continuous quantities (Vamvakoussi & Vosniadou, 2010). The researcher chose to investigate how parallel number lines supported students' reasoning regarding the relationship between fractions and decimals, because parallel number lines can simultaneously depict both fraction and decimal quantities and students can thus see when a fraction and decimal are equivalent.

Virtual Manipulatives

Manipulatives are a class of external representations of mathematical concepts investigated by mathematics education researchers. A manipulative is any object used to represent a mathematical concept that allows a student to interact with and manipulate the object in ways illustrating salient aspects of the represented mathematical concept. The reason often given for the instructional use of manipulatives is that they provide students with opportunities to learn mathematical concepts by physically interacting with representations of the mathematical concepts (Carbonneau, Marley, & Selig, 2013). Virtual manipulatives are a common type of manipulatives, implemented in computer-based learning environments. Indeed, Moyer-Packenham and Bolyard (2016) define a virtual manipulative as "an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" (p. 5).

Moyer-Packenham and Westenskow (2013, 2016) found that virtual manipulatives (VMs) have five specific categories of affordances for facilitating mathematics learning. These five categories of affordances are: focused constraint, where VMs constrain students' attention to specific intended features; creative variation, where VMs promote the variety and creativity of students' work; simultaneous linking, where different types of representations are linked with each other and with students' work; efficient precision, where VMs contain precise representations for efficient use; and motivation, where VMs motivate students to persist in mathematical tasks. In investigating the categories of affordances of virtual manipulatives, the researcher focused on features within the virtual manipulatives that were part of those affordance categories and how those features afforded students' reasoning regarding the relationship between fractions and decimals. The use of the term affordances in this study is directly aligned with Moyer-Packenham and Westenskow's description of affordance categories of virtual manipulatives.

Research demonstrates that VMs are effective for facilitating students' meaningful learning of fraction concepts. Moyer-Packenham and Westenskow (2013), in a meta-analysis of the effectiveness of VMs, observed that researchers conducted many studies of the effectiveness of VMs for instruction in the domain of fractions. Moreover, Moyer-Packenham and Westenskow found VMs used for fraction instruction had a moderate effect size of 0.53 over other forms of fraction instruction. Reimer and Moyer (2005) observed that VMs facilitated students' awareness of their misconceptions about fractions. However, an extensive search of the research literature revealed two gaps concerning students' learning of rational number concepts from virtual manipulatives. The first gap concerns the lack of research on students' learning of fraction and decimal concepts from number line-based VMs. The research by Steffe and colleagues is the most significant source of research making use of computer-based tools for the purposes of facilitating students' construction of rational number knowledge. Steffe and colleagues found that students are able to construct knowledge of fractions from the computer-based tools used in their studies (Olive & Lobato, 2008). These computer-based tools are based on the measure subconstruct of fractions and depict fractions as lengths, a conceptualization of fractions related to number lines (Steffe & Olive, 2010). However, the computer-based learning environments of Steffe and colleagues do not explicitly incorporate number lines.

The second gap is researchers have conducted very little research on students' learning of decimal concepts, as well as the relationship between fractions and decimals, using number line-based VMs. An extensive search of the literature of students learning of rational numbers involving computer-based tools or VMs found only one study involving students' learning of decimals from a computer-based learning environment (Rittle-Johnson et al., 2001).

Summary

The research and theoretical perspectives described earlier yield a theoretical framework useful for investigating students' reasoning about the relationship between fractions and decimals while using number line-based VMs. In particular, a learning environment incorporating constructivist tasks and number line-based VMs would be useful for investigating students' understanding of the relationship between fractions and decimals for fractions with terminating decimal representations. Number line representations are useful for investigating students' understanding of and reasoning regarding the relationship between fractions and decimals, because number lines can simultaneously depict both fractions and decimals.

Furthermore, as students engage in a series of tasks regarding the decimal-fraction relationship involving number line representations, they form mental models of the concepts. Students' reasoning for solving tasks provides clues about their mental models, which researchers can observe and document. These mental models may constitute flawed or incomplete knowledge, in the sense of Vosnidou's synthetic models.

Therefore, this literature review suggests the necessity of a study to answer the following research questions:

1. What are fifth-grade students' conceptions of fractions as decimals and decimals as fractions, for fractions with terminating decimal representations?

   1a. What synthetic models do students construct regarding the relationship between fractions and decimals, while working on tasks involving number line-based virtual manipulatives?

   1b. What is the evidence of students' reasoning about the relationship between fractions and decimals for fractions with terminating decimal representations?

2. What are the affordances of number line-based virtual manipulatives for supporting students' reasoning about the relationship between fractions and decimals as indicated by their hand gestures, mouse cursor motions, and explanations?

CHAPTER III

METHODOLOGY

Research Design

This study was conducted using a qualitative methodology, in which the researcher used clinical interviews and microgenetic methods to collect and analyze data (Chinn & Sherin, 2014). Effectively implemented clinical interviews are able to reveal information about how students construct knowledge, their cognitive processes, and their interpretations of learning situations and tasks (Ginsburg, 1997). Microgenetic methods allow for the detailed analysis of students' reasoning, particularly for research designs incorporating clinical and task-based interviews (Chinn & Sherin, 2014; Siegler 2006).

Participants

The subjects of this study were four fifth-grade students chosen from a local elementary school. The researcher considered fifth-grade students as ideal for the study, since, by the fifth-grade, students have typically acquired only a rudimentary knowledge of rational numbers in the form of fractions and decimals. Because of their learning from typical curriculum materials, fifth-grade students often know very little about the relationship between fractions and decimals. In addition, fifth-grade students have a very limited understanding of how to represent fractions and decimals using number line representations, even though students at this age are capable of learning from this type of representation (Moss & Case, 1999; Siegler et al., 2010). Furthermore, numerous studies indicate that students this age and younger are capable of explaining their reasoning and understandings concerning rational numbers when prompted by researchers (Moss & Case, 1999; Behr, Wachsmuth, Post, & Lesh, 1984; Steffe & Olive, 2010; Mack, 1990; Mack, 1995).

The researcher initially planned to gather data from six participants from a single fifth-grade classroom. However, the researcher was able to obtain IRB consent for only five participants, two boys (Dan and Rick) and three girls (April, Christy, and Lisa) (the names are pseudonyms to protect the privacy of the participants). Two criteria concerning the participants' responses on the initial clinical interview were to be used to determine their participation in the five task-based interviews and the final clinical interview. The first criterion was that it was necessary that the participants should be able to effectively express their ideas and reasoning verbally. The second criterion was that the participants should have a good understanding of both fractions and decimals: including knowledge of how to represent fraction and decimal quantities using part-whole representations, understanding of fraction equivalence, and understanding of the place-value structure of decimals. After administering the initial clinical interview to each of the five participants, the researcher determined that they each satisfied the above criteria. However, one participant (Rick) proved uncooperative during the task-based interviews and the researcher was not able to obtain a complete data set for this participant. Thus, the researcher was able to gather a complete data set for the four participants April, Dan, Christy, and Lisa.

Materials

The number line-based virtual manipulatives in this study were GeoGebra applets, which the researcher created. Virtual manipulatives in the form of applets can be embedded into a webpage and displayed using a web-browser (Moyer, Bolyard, & Spikell). This section presents an overview of the GeoGebra applets the researcher used in this study during the task-based interviews.

Researcher-Created GeoGebra Applets

GeoGebra is a powerful tool that allows users to create dynamically linked number line representations of fractions, decimals, and the decimal-fraction relationship. In particular, GeoGebra has a number of features allowing users to create applets incorporating dynamic linking (Moyer-Packenham & Westenskow, 2013), where students can see how changing the value of a fraction or decimal affects the location of a corresponding point on a number line, and how changing the point on a number line affects symbols for the corresponding fractions and decimals.

Because representing fractions and decimals as quantities or magnitudes is known to facilitate students' conceptual understanding of properties of fractions and decimals (Siegler et al., 2009), each of the GeoGebra applets used in this study emphasized the representation of fractions and decimals as lengths as well as points on a number line. There are a number of reasons for representing fractions and decimals as lengths. According to Clements and Sarama (2007) and Lehrer (2003), fifth-grade students are likely to have a well-developed understanding of the properties and uses of length for measurement and mathematical purposes. For instance, children at a young age are capable of understanding that lengths are useful for representing quantities, and understand that a greater length represents a larger quantity (Lehrer, 2003). Consequently, young children are also able to use lengths to compare the lengths of objects, and understand that larger objects have greater length measurements. Children develop these understandings from having engaged in measurement activities. Children also come to understand lengths as being composed of iterated unit lengths, and can understand the role of units in measuring lengths. In particular, children typically understand the inverse relationship between size of the unit and the number of units in a measurement, where more units are needed to measure a given length when the measurement unit is smaller. Because many children have well-developed ideas about lengths as quantities and the role of measurement units in measuring lengths, the researcher developed the GeoGebra applets to build on fifth-grade students' understanding of length by representing fractions and decimals using a length model. Two other reasons for depicting fractions and decimals as lengths is that measurement is a prominent interpretation of rational numbers (Behr, Harel, Post, & Lesh, 1992), and both fractions and decimals can be visually depicted as lengths (Fosnot & Dolk, 2002).

The screenshot shown below shows a task in which a student must construct the fraction to represent a given decimal (on the upper number line), where the decimal is presented as a length, and in this case the given decimal is 0.7. In this task, the student uses the sliders shown on the bottom to construct a fraction on the lower number line to match the length given on the upper number line.

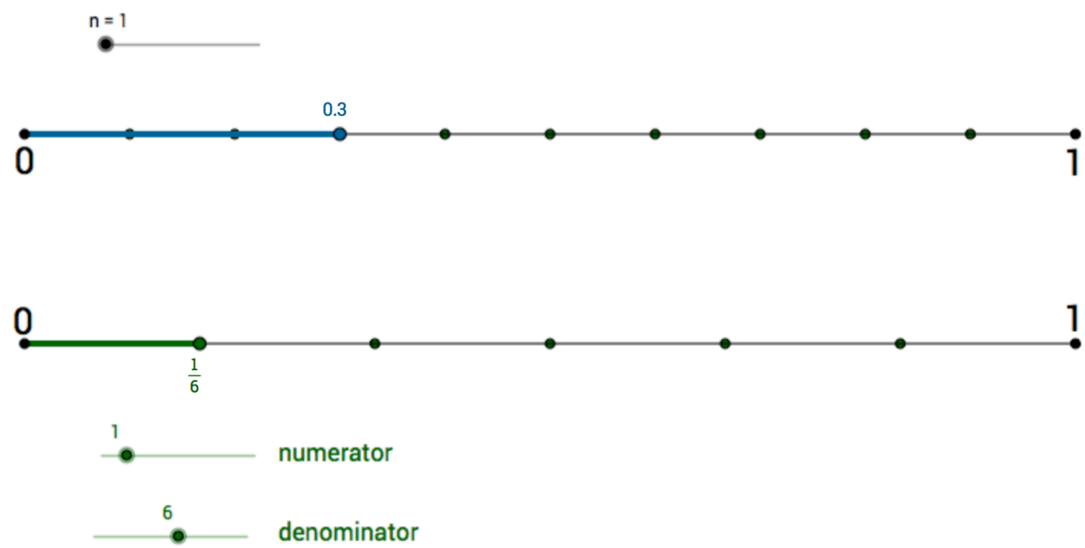

Figure 1. Screenshot of a GeoGebra applet in which students must construct a fraction and length to match the length of a given decimal.

Note that in the screenshot shown above, for the upper number line the student must make sense of the length for the decimal in terms of a fraction, where the applet provides little information that a student can use to construct the fraction corresponding to the given decimal of 0.7. This is a common feature of the applets, where students are not provided with all of the information about either lengths or fraction and decimal symbolism, to facilitate students' sense making and meaningful learning. Another feature of this representation is the slider at the top. By moving the slider to the next number n = 2, the student is provided with another decimal and length from which to construct a fraction and length to match. Students can repeat the process in this applet and construct fractional quantities for a total of 10 distinct given lengths.

The researcher created applets to elicit students' reasoning regarding the relationship between fractions and decimals, where fractions and decimals are represented as points and lengths on number lines. Furthermore, because of the fundamental importance of students' understanding proper fractions (fractions whose value is less than 1) (Booth & Newton, 2012), the fraction and decimal quantities used in the GeoGebra applets were restricted to the interval from 0 to 1.

Four Applet Types

During the task-based interviews, the students used three types of conversion applets and an applet for fraction and decimal comparison. The three types of conversion applets were *dual construction*, *one-way labeled*, and *one-way unlabeled*. When using the *dual construction* applets, students were prompted to make a conversion between fractions and decimals, and to use the sliders of the applets to construct both the fraction and decimal quantities as points and lengths on the parallel number lines. Figure 2 below shows a screenshot from a *dual construction* applet, which prompts students to convert fractions to decimals, where they used the sliders to make both the fraction and decimal as points and lengths on the number lines. The task depicted in the screenshot is to find the fraction equivalent of 0.05, which is 1/20, and to use the sliders to make both 0.05 and 1/20.

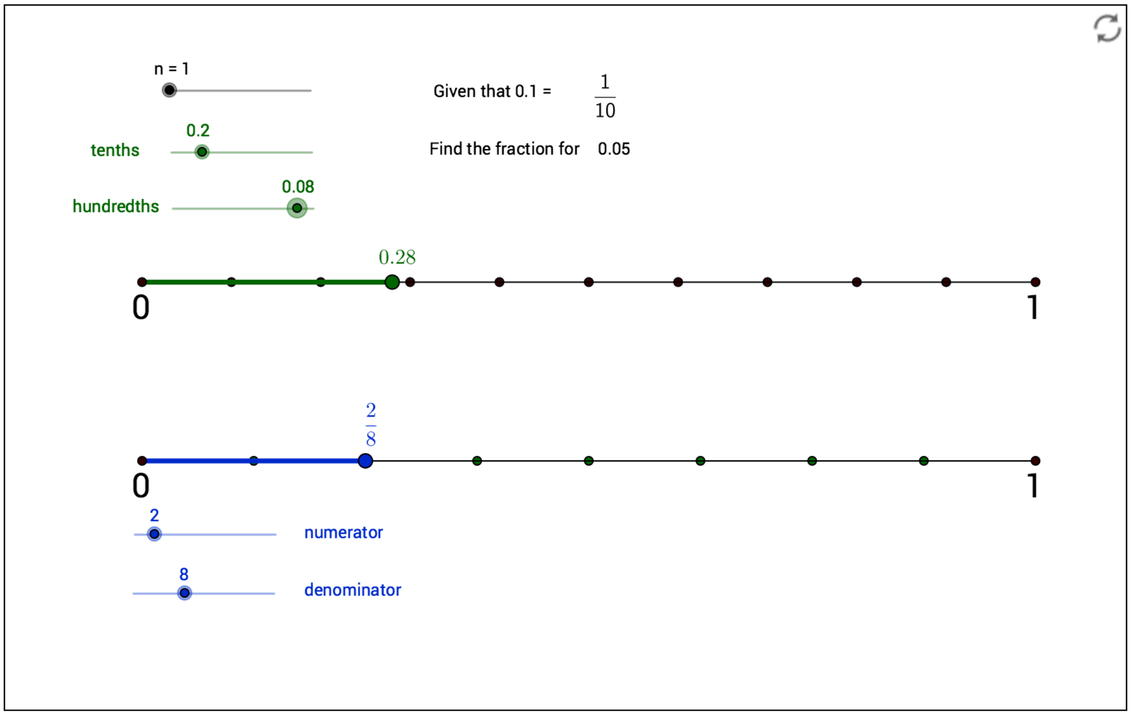

Figure 2. Screenshot of a dual construction applet.

One-way labeled was the second type of applet used to present conversion tasks to the students. *One-way labeled* applets presented students with a number to be converted as well as its corresponding point and length on the upper number line. The task for students was to convert the given number to the *target number type* and then use the sliders to construct the target number and segment on the lower number line. Figure 3 below shows a screenshot of a *one-way labeled* applet. The specific task depicted in the screenshot is to determine the fraction equivalent of 0.85 and to use the sliders to make the fraction equivalent, which is 17/20.

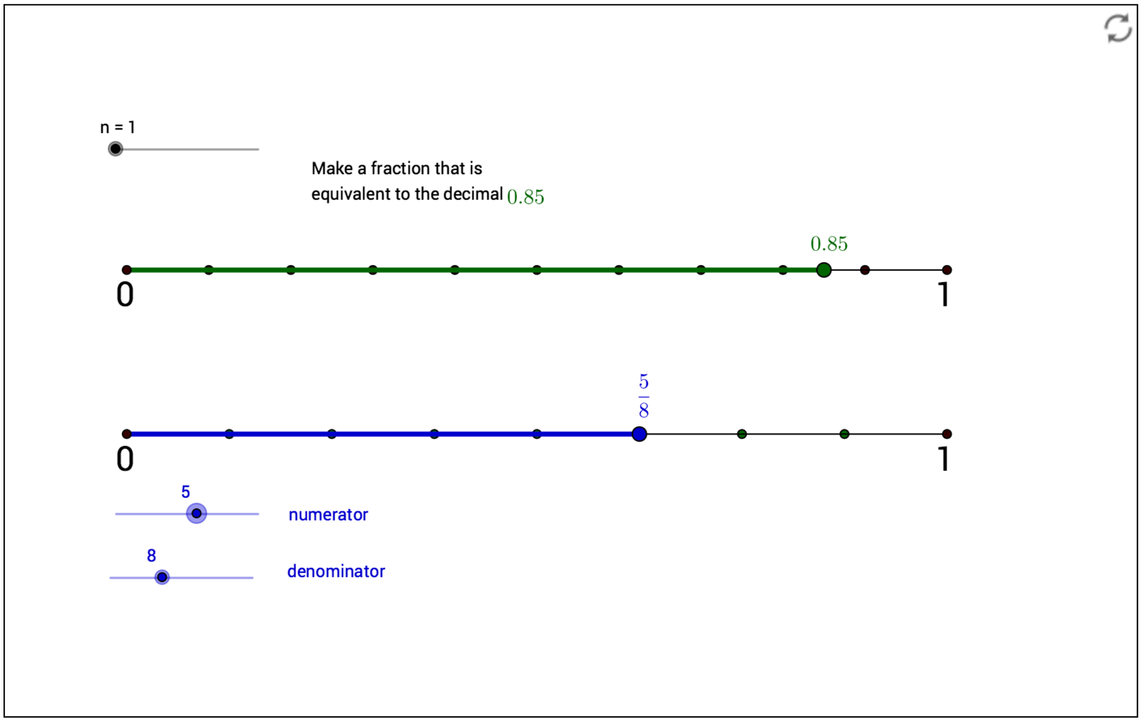

Figure 3. Screenshot of a one-way labeled applet.

One-way unlabeled was the third type of applet students used during tasks of converting between fractions and decimals. *One-way unlabeled* applets presented students with an unlabeled point and length on a number line, where it was first necessary to interpret the displayed point and length as either a fraction or a decimal. After determining the quantity represented by the point and length, the student needed to convert this quantity to either a fraction or a decimal, depending on whether the unlabeled quantity is a decimal or a fraction. Figure 4 below shows a screenshot of a *one-way unlabeled* applet. In the screenshot, the applet prompts students to interpret a given point and length as a fraction (which in this case represents the fraction 11/20) to determine the decimal equivalent to 11/20, and to use the sliders to construct the decimal equivalent of 11/20.

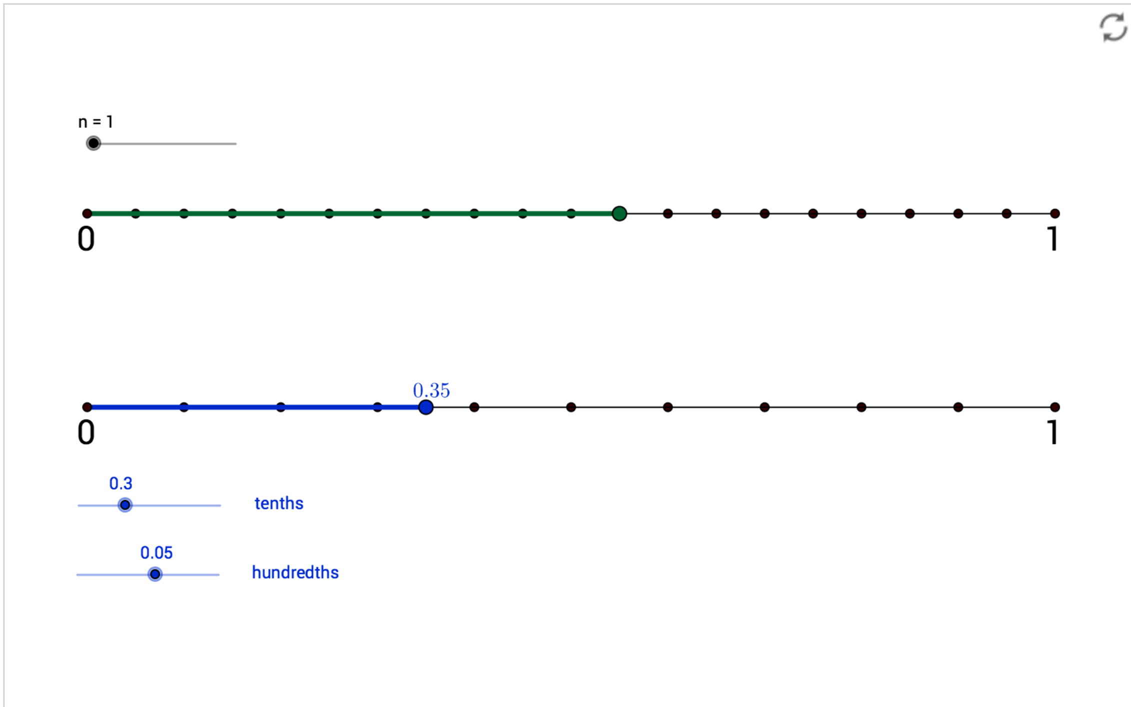

Figure 4. Screenshot of a one-way unlabeled applet.

The *comparison* applet was the fourth type of applet students worked with during the task-based interviews. The *comparison* applet presented students with pairs of fractions and decimals, and prompted them to determine for each pair which quantity was the larger for each pair. After stating which quantity was largest and explaining why, students then used the applet to make both the fraction and decimal. Figure 5 below shows a screenshot from the *comparison* applet. In the screenshot, the applet prompts students to determine which is larger of ¼ and 0.85, and then to use the sliders to make the point and length for each quantity on the two number lines.

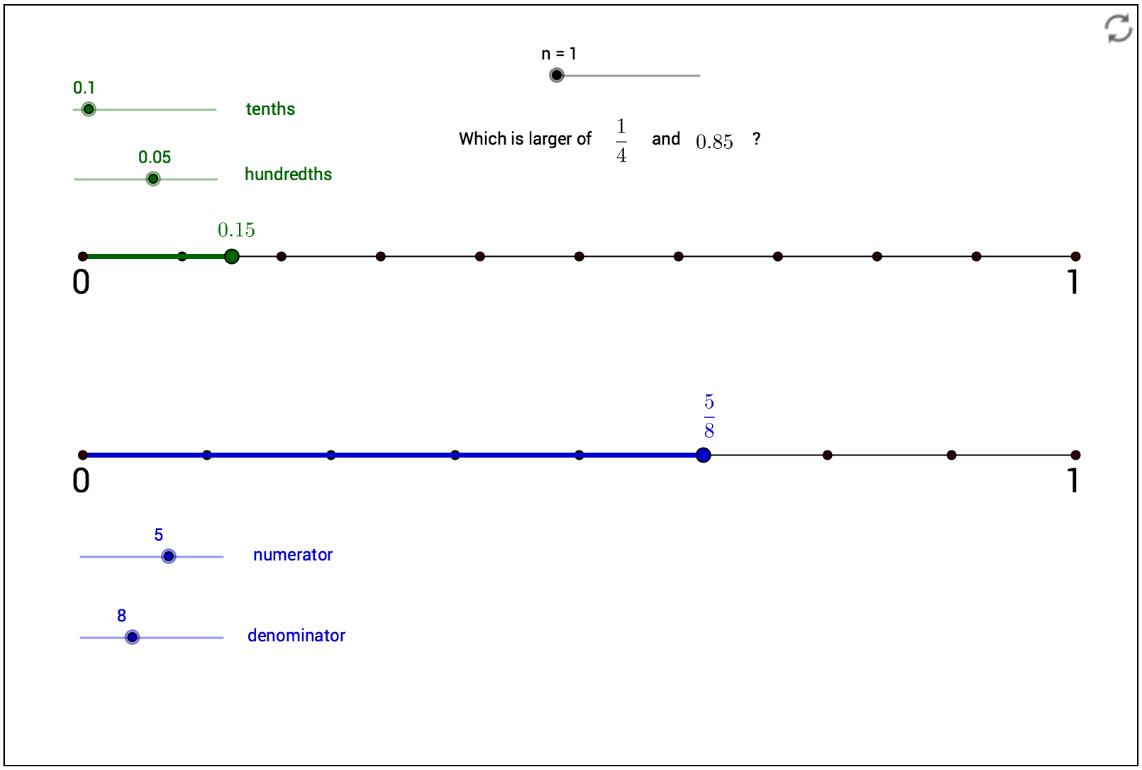

Figure 5. Screenshot of the comparison applet.

GeoGebra Applets' Alignment with the Conceptual Stepping Stones

The constructed applets support the conceptual stepping-stones described in the previous chapter, including the unit or whole for both fractions and decimals; unit numbers for both fractions and decimals; non-unit and benchmark fractions for both fractions and decimals; partitioning; and iterating unit numbers to create non-unit numbers.

Each of the applets the researcher created for this investigation was designed to support students' understanding of fractions and decimals as quantities or magnitudes, since the applets depict fractions and decimals as lengths.

Because each of the applets were based on the unit segment from 0 to 1, and the endpoints of the segments were clearly labeled, the relevant *unit* or *whole* was always apparent to the students. Because of this feature, the applets in this study supported the students' understanding of the relevant unit or whole.

A third conceptual stepping stone emphasized the importance of unit fractions and decimals as a necessary foundation for the students' understanding of the decimal-fraction relationship. The researcher implemented specific tasks incorporating the applets in which the students created the decimal equivalent for given unit fractions, thus supporting students' understanding of the decimal-fraction relationship for unit fractions.

The researcher created two applets designed to support students' understanding of the decimal-fraction relationship for benchmark numbers. The first applet prompted students to construct decimal equivalents for given benchmark fractions, and the second prompted students to construct fraction equivalents for given benchmark decimals.

Each of the GeoGebra applets used in this study supported students' understanding of partitioning the unit or whole. For instance, number lines used depict fraction quantities were partitioned according to the denominator of the represented fraction. Similarly, number lines used to depict decimal quantities were partitioned into ten sub-segments.

The basic functionality of the sliders supported students' iteration of unit fractions and decimals. The researcher designed activities that asked students to create fractions and decimals by iterating unit fractions and decimals, in which students were prompted to observe the relationship between the lengths of the iterated number and the original unit number.

The researcher created specific applets in which students were prompted to compare fractions and decimals. For instance, in one applet, students were prompted to construct pairs of fractions and decimals and then compare the numbers. Furthermore, because the applets depict the fractions and decimals as lengths, this feature contributed to students' comparison of fraction and decimal quantities.

A number of applet-based activities incorporated halving and doubling activities for decimal-fraction combinations. In one type of activity, students were prompted to construct half of a given decimal-fraction combination, and a second type of activity prompted students to make new decimal-fraction combinations by doubling the decimal fraction combination for given numbers.

Procedures

Implementation of the Study

Data gathering for this study consisted of three phases: an initial clinical interview, five task-based interviews involving the GeoGebra, and a final clinical interview. The researcher worked with each of the students for approximately 7-10 school days near the end of the 2014-2015 academic school year. Table 1 below shows the timeline for the study.

Table 1

Timeline of the Study

Event	Activities
Initial Clinical Interview	Clinical Interview
First Task-Based Interview	Conversion of decimals to fractions with denominators of 10 (0.1, 0.3, 0.5, 0.7, etc.) Conversion of fractions with denominators of 10 to decimals (1/10, 3/10, 5/10, etc.) Conversion of fractions with denominators of 100 to decimals (31/100, 49/100, 63/100, etc.) Conversion of hundredths fractions to decimals (37/100, 57/100, 87/100, etc.)
Second Task-Based Interview	Conversion of fractions with denominators of 5 to decimals (1/5, 2/5, 3/5, etc.) Conversion of decimals to fractions with denominator of 5 (0.2, 0.4, 0.6, etc.)
Third Task-Based Interview	Conversion of fractions with denominators of 20 to decimals (1/20, 2/20, 3/20, etc.) Conversion of decimals to fractions with denominators of 20 (0.05, 0.15, 0.35, etc.) Conversion of fractions with denominators of 25 to decimals (1/25, 2/25, 3/25, etc.) Conversions of decimals to fractions with denominators of 25 (0.04, 0.08, 0.44, 0.88, etc.)
Fourth Task-Based Interview	Conversion of fractions with denominators of 8 to decimals (1/8, 3/8, 5/8, etc.) Conversion of decimals to fractions with denominators of 8 (0.375, 0.625, 0.875, etc.)
Fifth Task-Based Interview	Comparison of fractions and decimals
Final clinical interview	Clinical interview

Initial clinical interview. At the beginning of the study, the researcher administered to the four participating students an initial clinical interview that lasted approximately 45 minutes. The initial clinical interview was conducted with fifth-grade students to select participants for the task-based interviews, and to gauge participants' initial understanding of decimals, fractions, and their relationship, and participants' understanding of how to represent fractions and decimals on number lines. The purpose of the interview was to determine each student's knowledge of fractions and decimals, including their misconceptions, knowledge of the decimal-fraction relationship, and to assess their understanding of locating fractions and decimals on number lines. Specifically, the researcher assessed students' performance on several tasks involving fractions, decimals, and number lines. Knowledge specifically assessed during the clinical interviews included: students' understanding of how to locate benchmark numbers of fractions and decimals on number lines; ordering tasks for fractions and decimals; tasks involving equivalent fractions; and, understanding of the relationship between fractions and decimals for benchmark numbers. Another purpose of the clinical interview was to establish rapport with each of the students. During the initial clinical interview, the researcher provided each of the participants with pencil and paper in case participants wished to use these resources for computations or to make any drawings related to the given fraction and decimal tasks. Participants' hand written drawings and computations were logged and used by the researcher to inform the analysis of the data.

Task-based interviews. Five task-based interviews followed the initial clinical interview. During the five task-based interviews, the students engaged in tasks incorporating the GeoGebra applets designed to elicit their reasoning regarding the decimal-fraction relationship. The GeoGebra applets incorporated two parallel number lines, where the first number line represented fraction quantities, and the second represented decimal quantities.

The researcher used the task-based interviews to elicit the students' reasoning regarding the decimal-fraction relationship for various types of fractions with terminating decimal representations. In the first task-based interview, students used the GeoGebra applets to perform conversions between fractions and decimals for fractions with denominators of 10 and 100. In particular, students used the applets to convert fractions with denominators of 10 to fractions and convert decimals to fractions with denominators of 10. The students also used the applets to convert fractions with denominators of 100 to decimals and to convert decimals to fractions with denominators of 100. The first task-based interview with each participant lasted approximately 40 minutes.

During the second task-based interview, students completed tasks using the GeoGebra applets to make conversions between fractions and decimals for fractions with denominators of five. Students used the applets to convert fractions with denominators of five to decimals, and to convert decimals to fractions with denominators of five. The second task-based interview was the briefest for each of the participants, taking approximately 25 minutes.

In the third task-based interview, the researcher presented the students with tasks involving conversions between fractions and decimals for fractions with denominators of 20 and 25. Students used the applets during conversions of fractions with denominators of 20 to decimals and during conversions of decimals to fractions with denominators of 20. In addition, students also used the applets during tasks of converting fractions with denominators of 25 to decimals and during tasks of converting decimals to fractions with denominators of 25. The third task-based interview was the longest for each of the participants, lasting approximately 45 minutes.

During the fourth task-based interview, students engaged in tasks of converting between fractions and decimals for fractions having denominators of eight. In particular, the students used the applets to perform several tasks of converting fractions with denominators of eight to decimals and tasks of converting decimals to fractions with denominators of eight. The researcher worked with each of the participants for approximately 30 minutes during the fourth task-based interview.

In the fifth task-based interview, students used a GeoGebra applet to compare fractions and decimals involving fractions with denominators of 5, 20, 25, and 8 or the decimal equivalent of fractions with these same denominators. Students were able to complete all of the comparison tasks during the fifth task-based interview in approximately 30 minutes.

To gain information about the features of the applets that afforded opportunities for students' reasoning regarding the decimal-fraction relationship, the researcher frequently prompted students during the task-based interviews to explain how they used the applets during the tasks involving the relationship between fractions and decimals. An example of a specific prompt is "Can you tell me how these number lines are helping you to solve this problem?" During the coding phase of the study, the researcher analyzed students' responses to such prompts, to determine if they provided information about affordances of the applets for supporting students' reasoning regarding the decimal-fraction relationship. By reflecting on and taking notes after each task-based interview, the researcher developed what Chi (1997) refers to as "impressions" (p. 281) of each student's understanding of and reasoning regarding the decimal-fraction relationship. The researcher verified the documented impressions during the data analysis phase of the study, and used the documented impressions to form initial coding categories during data analysis.

During the task-based interviews, the researcher provided each of the participants with pencil and paper in case participants wished to use these resources for computations or to make any drawings related to the given fraction and decimal tasks. The researcher logged participants' use of these materials, which were subsequently used to inform the analysis of the data.

Final clinical interview. At the conclusion of the study, the researcher conducted the final clinical interview with each student. This clinical interview involved tasks and activities similar to those used in the initial interview. The purpose of the second clinical interview was to determine if any changes occurred in each of the student's reasoning regarding the relationship between fractions and decimals from the initial clinical interview. As was the case with the other interviews, the researcher provided participants with pencil and paper in case they wished to use these resources. Participants' use of these materials was used to inform the data analysis. Participants finished the final clinical interview in approximately 45 minutes.

Data Sources and Instruments

This study was based on three primary data sources: video recordings of clinical interviews and task-based interviews; video recordings of students' facial expressions from a computer webcam; video recording screen captures of students' use of the GeoGebra applets. In situations where students interact with a computer during data gathering sessions, Lesh and Lehrer (2000) recommend recording students from two perspectives. The use of two video sources during the data collection allowed the researcher to accurately transcribe nearly 100% of participants' verbal statements. Video files from the recorded initial and final clinical interviews and task-based interviews were stored on a computer hard drive, as well as on a portable external hard drive.

Video Recordings of Clinical Interviews and Task-Based Interviews

For the initial and final clinical interviews, a single video camera was used to record students' explanations, gestures and actions. During these interviews, students were seated at a table, the researcher was seated opposite the student, and the video camera was located nearby on a tripod perpendicular to the student and facilitator, to record the words and actions of both student and facilitator.

During the task-based interviews, students were engaged in tasks while interacting with the web-based GeoGebra applets from a laptop computer. During these interview sessions, a video camera recorded the interviews from nearby on a tripod, located slightly to the side of where the students were seated, and captured the students' use of the laptop computer as well as their gestures.

Video Recordings of Facial Expressions from the Laptop Webcam

During the task-based interviews, the researcher used Screenflow to record students' facial expressions using the laptop computers' built in webcam as the students used the GeoGebra applets. Screenflow also recorded the students' verbal explanations during these sessions.

Screen Captures of Students' Use of the Applets

The researcher used Screenflow to record students' interaction with the GeoGebra applets and mouse cursor behavior during the task-based interviews.

Pilot Testing of Interview Instrument

Before the study began the researcher pilot tested a clinical interview instrument by interviewing 12 students from grades 4 to 7 using an instrument developed by the researcher, which can be found in Appendix B.

During interviews, the researcher used the clinical interview instrument to probe students' knowledge of several topics related to their understanding of fractions and decimals. For fractions, the questions probed students' understanding of order and equivalence, how they mentally represent fractions, and their understanding of fractions as quantities or magnitudes. Regarding decimals, the questions probed students' understanding of order and place value properties, how they mentally represented decimals, and their sense of the quantities represented by decimals. The questions also probed students' understanding of the relationship between fractions and decimals, and their understanding of how to represent fractions and decimals on number lines.

The clinical interview involved a number of tasks for fractions, decimals, and number lines. Many of the tasks had students construct or order fractions and decimals from numerals and fractions printed on card stock. For the number line tasks, the researcher presented students with fractions or decimals, and the researcher prompted the students to indicate where the numbers were located on a large number line.

Pilot testing the interview instrument allowed the researcher to refine the tasks and prompts, as well as to eliminate some tasks that seemed to be too difficult for the students or that yielded responses of little interest. By conducting the pilot interviews, the researcher gained valuable practice in asking students to clarify their responses in ways not too demanding or intrusive for students, and which revealed details about their understanding and reasoning. Additionally, by conducting the pilot interviews, the researcher gained valuable ideas concerning the design and implementation of tasks for this study. Furthermore, the pilot interviews allowed the researcher to thoroughly test the video equipment and system for archiving video files.

Data Analysis

Coding and Analysis of Students' Verbal Data

First, the researcher transcribed all of the interviews into text form for analysis and coding. Next the researcher used the constant-comparative method (Glaser & Strauss, 1967) to analyze the data.

Constant comparative method of data analysis. The researcher analyzed the data in four stages (Glaser & Strauss, 1967). First, the researcher coded the data from verbal transcripts and two video sources. Incidents within the data were coded into categories, and incidents within the categories were compared to define resulting categories. A key part of the constant-comparative method is memo writing. Because memo writing is a key part of the constant-comparative method, while coding data, the researcher stopped to record memos pertaining to the emerging coding categories, to make notes about the creation of new categories, to adjust existing categories, as well as to generate theory about the relationships among categories.

During the second stage of the constant-comparative analysis, the researcher integrated the emerging categories and their properties. The researcher undertook this by comparison of categories, in addition to the comparison of incidents within distinct categories. This aided in the delimitation of the emerging categories. While undergoing this process, the researcher made theoretical sense of the comparisons of separate categories, which contributed to the emerging theoretical constructs.

The third stage of the constant comparative analysis resulted in the refining of coding categories and the delimitation of the emerging theoretical constructs. The researcher only included categories relevant to the emerging theory, and discarded any others. During this stage of the analysis the theory became increasingly definitive and theoretically saturated in the sense that further coding did not produce additional categories, where, the researcher identified a smaller number of concepts relevant to the theory, in order to achieve parsimony in the emerging theory. Consequently, there were fewer and fewer major modifications to the emerging theory during this stage of the analysis.

By the fourth stage of the constant-comparative analysis, the researcher had fully specified the theoretical constructs that emerged from the data and was able to use data to support those theoretical constructs.

Coding

Coding of synthetic models. The researcher coded the students' explanations of mathematical reasoning for the purpose of characterizing the students' mental models regarding the relationship between the relevant fractions and decimals. To document the students' synthetic models, the researcher identified the coded explanations of reasoning that reflected mathematically inaccurate reasoning regarding the decimal-fraction relationship.

Coding of conversion mathematical operations and strategies. In order to answer Research Question 1a, regarding students' conceptions of the relationship between fractions and decimals, the researcher followed a two-stage coding process during this phase of the coding. During the first stage of the coding, the researcher identified and coded any mathematical operations the students mentioned during explanations of conversions between fractions and decimals. Examples of such mathematical operations included addition, subtraction, multiplication, and division. Second, following the coding of the mathematical operations, the researcher coded the conversion strategies that students used to make conversions of fractions to decimals and decimals to fractions. The researcher considered a conversion strategy to be a method used by a student for the purpose of converting a fraction to a decimal or a decimal to a fraction, consisting of the application of component mathematical operations, such as multiplication, division, addition, or subtraction of quantities used by a student during an explanation of converting between fractions and decimals.

Coding of affordance related features through explanations, gestures, and mouse behavior. The coding and analysis that allowed the researcher to answer research question 2 regarding the features of the applets that afforded opportunities for students' conversion reasoning included the coding of students', hand gestures, and mouse cursor motions, and verbal explanations that students made that appeared to be related to properties of the applets that supported students' conversion reasoning.

To document evidence of what the app features afforded from students' gestures, the researcher coded gestures that indicated that students were attending to features of the applets that supported reasoning about the relationship between the relevant fractions and decimals. In particular, the researcher followed the conventions of Goldin-Meadow (2003) and coded gestures into the three categories of *deictic* gestures, *iconic* gestures, and *metaphoric* gestures. *Deictic* gestures are those gestures in which students use their hands to point or indicate something. Deictic gestures were coded since these types of gestures can aid in clarifying students' spoken words and explanations. Second, the researcher coded students' *iconic* gestures, which are gestures representing "body movements, movements of objects or people in space, and shapes of objects or people" (Goldin-Meadow, 2003, p. 7). The researcher coded *iconic* gestures, since they provide information about students' thoughts and reasoning (Goldin-Meadow, 2003). Third, the researcher coded students' *metaphoric* gestures. According to McNeill (2005), *metaphoric* gestures contain “images of the abstract,” (p. 39) and thus provide insights into students’ conceptions and thinking.

Additionally, the researcher coded motions of the mouse cursor made by students that indicated they were attending to features of the applets that supported their reasoning regarding the decimal-fraction relationship or conversions between fractions and decimals.

Furthermore, the researcher coded explanations made by students that indicated they were attending to features of the applets that supported their reasoning regarding the relationship between fractions and decimals.

At the conclusion of the coding of the students’ affordance-related explanations, gestures and mouse cursor motions, the researcher identified how the resulting affordances aligned with the five categories of affordances of virtual manipulatives of *focused constraint*, *creative variation*, *simultaneous linking*, *efficient precision*, and *motivation* identified by Moyer-Packenham and Westenskow (2013, 2016).

Analysis of Data

Table 2 below depicts the initial categories for the coding of students’ verbal data.

Table 2

Data Analysis for Data from Clinical Interviews and Task-based Interviews

Research Question	Data Source	Data analysis
1. What are fifth-grade students' conceptions of fractions as decimals and decimals as fractions for fractions with terminating decimal representations?	Transcript data	Transcript excerpts providing evidence of students' conceptions of fractions as decimals and decimals as fractions.
1a.What synthetic models do students construct regarding the relationship between fractions and decimals, while working on instructional tasks involving number line-based virtual manipulatives?	Transcript data	Transcript excerpts providing evidence of specific synthetic models of students
1b.What is the evidence of students' reasoning about the relationship between fractions and decimals for fractions with terminating decimal representations?	Transcript data	Identification of strategies used by students to convert fractions to decimals and decimals to fractions Frequency counts of students' strategies for converting fractions to decimals and decimals to fractions

2.What are the affordances of number line-based virtual manipulatives for supporting students' reasoning about the relationship between fractions and decimals as indicated by their hand gestures, mouse cursor motions, and explanations?	Transcript data from task-based interviews. Video recorded data of students' gestures during task-based interviews. Screenflow recordings of students' mouse cursor motions	Identification of affordances of applets supporting students' conversion reasoning as indicated by students' explanations Frequency counts of students' affordance-related explanations, gestures, and mouse cursor motions Bar charts of affordance-related explanations, gestures, and mouse cursor motions

Triangulation of data. The researcher established the validity of analyzed and interpreted data through the triangulation of data from multiple sources, a process that results in convergent validity (Ginsburg, 1997). According to Ginsburg, using multiple data sources to establish convergent validity is an effective approach for establishing the validity of students' verbal data, such as the verbal data arising from clinical interviews. The researcher was able to use multiple data sources to triangulate data regarding participants' verbal statements, hand gestures, and mouse cursor motions while they used the applets. The multiple data sources the researcher analyzed included three sources of video data.

CHAPTER IV

RESULTS

The researcher conducted this dissertation study for two closely related purposes: first, to investigate a sample of fifth-grade students' reasoning regarding the relationship between fractions and decimals for fractions with terminating decimal representations while using virtual manipulative incorporating parallel number lines; second, to investigate the affordances of the virtual manipulatives for supporting the students' reasoning about the decimal-fraction relationship, using the categories of affordances identified by Moyer-Packenham and Westenskow (2016) specifically for virtual manipulatives.

Research Question 1a: Students' Synthetic Models

This section addresses Research Question 1a regarding the students' synthetic models of the decimal-fraction relationship. In particular, this section answers Research Question 1a by presenting transcript data as evidence of synthetic models from the two students April and Christy as they completed tasks involving the relationship between fractions and decimals while using the number line applets. In particular, these two students held similar synthetic models that were observed during two types of tasks: finding the decimal equivalent of 1/8; and particular fraction and decimal ordering tasks. Furthermore, these students' synthetic models interfered with their ability to execute these two types of tasks. This section begins by describing April's synthetic model, followed by a description of Christy's synthetic model.

April's Synthetic Model during the 1/8 Task

The following transcript excerpt taken from the fourth task-based interview with April begins with April attempting to find the decimal equivalent of 1/8. As April reasons about finding the equivalent decimal, she realizes that the task involves taking half of 25/100, and correctly states that the resulting fraction is 12.5/100. However, at this point April provides evidence of a synthetic model by repeatedly expressing her belief that 12.5/100 is an incorrect way of expressing a fraction, which by her reasoning is incorrect since the numerator contains a decimal:

001	Interviewer	Okay, so you're going to figure out one eighth is exactly, is that what you're going to do?
002	April	[April uses paper and pencil in an attempt to determine the decimal for 1/8]
003	April	That wouldn't really work. So, you times that by eight it would equal a hundred, but that wouldn't really work.
004	Interviewer	So, that's kind of a…so you've got twelve and a half there, so I'm kind of wondering where you get that from, because you're really in the neighborhood.
005	April	Well, I know it won't really work because you can't have a decimal as a decimal, like a decimal for a fraction which that's pretty much what a decimal is. But, what I got is four, so that's twenty five times four equals a hundred, so if I make that an eight, I could split that in half, because four is half of eight, and that's twelve point five. But you can't make twelve point five decimal, like...
006	Interviewer	Okay, so, you have some good ideas here. If you knew the decimal for one fourth, do you think you could find the one for one eighth?
007	April	Yeah, but, that includes a decimal, with the decimal, which you can't do.
008	Interviewer	I see.
009	April	You can't do a decimal, like, over a fraction.
010	Interviewer	So, how does this relate to one fourth, though?
011	April	Because, one fourth equals point twenty five. So, if want to find one eighth, I'd need to find the decimal that eight times what would equal one hundred. So, I could

		also do a hundred divided by eight, maybe that would get a different answer, but...
012	April	[April uses paper and pencil to divide 100 by 8]
013	April	Yeah, it's twelve point five, but that wouldn't work, because, again you can't do a decimal on a fraction.

In the last sentence in line 005 of the transcript excerpt, April speaks of being unable to use 12.5. What she means by this is that in her attempt to make a fraction equivalent to 1/8 with a denominator of 100, she notices that the numerator of the equivalent fraction must be 12.5, and she refuses to acknowledge this as a legitimate value for the numerator of a fraction. In line 007, April again reiterates the inappropriateness of having a decimal in the numerator of a fraction. Furthermore, in line 009, April clarifies this same point by stating “You can’t do a decimal, like, over a fraction.” In line 011, April attempts to divide 100 by 8 to find the numerator of the fraction equivalent to 1/8 and with a denominator of 100, which she again finds must be 12.5. In line 013, she reiterates that she cannot do this because it is not permissible to use 12.5 in the numerator of a fraction.

Two conclusions can be drawn from the above transcript excerpt regarding April’s synthetic model of the relationship between fractions and decimals. First, April holds the belief that a fraction cannot properly be expressed in the form of a decimal divided by a whole number. Second, April’s synthetic model appears to have its basis in a weak understanding of fraction equivalence. April apparently did not consider the possibility of creating the equivalent fraction 125/1000 by multiplying the numerator and denominator of 12.5/100 by factors of 10. A possible reason for April’s difficulties during this task is that she may have had a limited understanding of the thousandth decimal place and that the decimal 0.001 is equivalent to the fraction 1/1000.

April's Synthetic Model during Two Ordering Tasks

In the following transcript excerpt, which was taken from the fifth task-based interview with April, she was presented with the task of ordering 4/5 and 0.45. In her approach to this task, April attempted to use the strategy of comparing both of these numbers with the benchmark value of ½. The transcript excerpt shows evidence of the same synthetic model demonstrated in the previous task of finding the decimal equivalent of 1/8. In this excerpt, she again expresses the inappropriateness of a fraction with a decimal in the numerator.

014	Interviewer	So, four fifths and point forty five, which one do you think is larger?
015	April	Well, without looking at it, I think four fifths is going to be larger, because, I just, forty five again is less than half, whereas five doesn't really have a half, but four would be larger than the half, because I guess three is sort of a half, but not really, but, if that makes sense.
016	Interviewer	Yeah, yeah, I mean, technically, if you wanted to get half of five would be what?
017	April	It would be three point five, but you can't do a decimal over a fraction, so that wouldn't work.

In line 015 of the above transcript, April attempts to find a fraction with a denominator of five that is equivalent to ½. Here, she claims that there is no fraction with a denominator of five that is equivalent to ½ by first stating that five does not have a half, and that "three is sort of a half, but not really." When prompted by the interviewer to state what half of five is, April states in line 017 of the transcript excerpt that 3.5 is half of 5, but then emphasizes that "you can't do a fraction over a decimal," and that 3.5/5 will not work as a fraction equivalent to ½. It is possible that because of limited proficiency with equivalent fractions, April did not convert 4/5 and ½ to equivalent fractions with denominators of ten, which would yield the fractions 8/10 and 5/10.

April also displayed evidence of the same synthetic model in a similar task during the same task-based interview as illustrated in the following transcript excerpt, in which she was presented with the task of ordering 2/5 and 0.25. However, during this task, instead of attempting to compare both quantities with the benchmark number ½, April chose to convert 2/5 to the decimal 0.40 so she could order 0.40 and 0.25 by using decimal place value. In this excerpt, which occurred during the fifth task-based interview with April, she exhibits evidence of a synthetic model by stating the difficulty of comparing fractions with odd denominators, such as five and nine, with the benchmark number ½:

018	Interviewer	Two fifths and point two five.
019	April	I think the one that's going to be larger is probably two fifths, because if you times five by twenty you could just make a decimal, that's point four.
020	Interviewer	Okay, so this one, the two fifths is point four?
021	April	Yeah, I didn't really compare these ones to a half, but, it's kind of hard to do it with fifths, and ninths and stuff. Yeah, two fifths.

In line 019 above, April converts 2/5 to the decimal 0.40 by multiplying the numerator and denominator of 2/5 by 20. Then, in line 021, April expresses that she deliberately did not attempt to compare 0.25 and 2/5 with ½, expressing the difficulty of comparing 2/5 with ½, consistent with the synthetic model of the inappropriateness of a fraction with a decimal numerator.

Christy's Synthetic Model during the 1/8 Task

When the researcher presented Christy with the task of finding the decimal equivalent to 1/8, she also exhibited evidence of the same synthetic model that April showed during the same task. Indeed, the following transcript excerpt reveals that Christy attempted to find the decimal for 1/8 by taking half of each of the two equivalent quantities 1/4 and 25/100. The following transcript excerpt, which occurred during the fourth task-based interview with Christy, reveals her hesitation to accept that half of 25 could be used as the numerator of a fraction:

022	Interviewer	So, one fourth, what is the decimal for that one? Do you know what that one is?
023	Christy	Yeah, point two five.
024	Interviewer	So, we know that one fourth is point two five. So, if we knew that then how would we find the decimal for one eighth?
025	Christy	Half point two five.
026	Interviewer	Half point two five. It seems like we've talked about this before, how, maybe your dad taught you how to divide a decimal.
027	Christy	Yeah, well, I just barely thought about that, so, I'm just guessing. Um, well, you can't really half twenty five, but...
028	Interviewer	Well, let's say halving twenty five, like the actual number twenty five.
029	Christy	It would be about twelve and a half.
030	Interviewer	Twelve and a half. So, can you do a similar thing with a decimal?
031	Christy	Uh huh.

In line 025 of the above transcript excerpt, Christy correctly reasons that the decimal equivalent of 1/8 is half of the decimal 0.25. However, in line 027, Christy expresses difficulty finding half of 0.25, because "you can't really half twenty five." After being prompted by the interviewer, in line 029 Christy expresses that half of twenty five is twelve and a half. In line 030, the interviewer then asks if she could similarly take half of 0.25. Evidently, because of this prompting, Christy realized how to make the decimal equivalent of 1/8, because she subsequently went on to use the sliders in the number line applet to construct the decimal 0.125.

Christy's Synthetic Model during an Ordering Task

During the following transcript excerpt, which occurred during the fifth task-based interview with Christy, she was presented with the task of ordering 4/5 and 0.45, the same task in which April displayed evidence of a synthetic model. In the transcript excerpt, Christy had already used the applet to make 4/5 and 0.45, and she initially uses visual evidence from the applet for justification that 4/5 is greater than 0.45. The interviewer then asks Christy to provide reasons why 4/5 is greater than 0.45, and she responds by attempting to apply the strategy of ordering the two numbers by comparing them with the benchmark number of ½, but expresses difficulty in finding a fraction equivalent to ½ with a denominator of five:

032	Interviewer	So, four fifths and point four five.
033	Christy	Um.
034	Interviewer	So, which one do you think is larger?
035	Christy	Probably four fifths.
036	Interviewer	How come?
037	Christy	Because this is four fifths (Christy points at 4/5) and point four five is like that (Christy points at the 0.45 she made on the decimal number line), so, yeah.
038	Interviewer	So, what would be a reasoning that you would have for that? Why...?
039	Christy	Well, point, point, see, yeah, point four five is closer to one half, and then four fifths, or like five, you divide a piece into five they don't really, it doesn't really have a half, so, yeah.
040	Interviewer	Oh, okay, so yeah there's not half...

041	Christy	A half.
042	Interviewer	...doesn't, isn't a dot on that? Okay.
043	Christy	Yeah.

In line 037, Christy refers to the applet for justification of why 4/5 is greater than 0.45. In line 038, the interviewer responds by asking Christy to provide reasons for why 4/5 is greater than 0.45. In line 039, Christy attempts to order 0.45 and 4/5 by comparing both with the benchmark number 1/2. However, she goes on to express the idea that it is not possible to take half of a whole that is divided into five equal pieces. Furthermore, in line 039, when Christy states "it doesn't really have a half," she appears to mean that there is no fraction with a denominator of five that is equivalent to 1/2. In a manner similar to that of April, Christy does not consider converting 4/5 and 1/2 to equivalent fractions with the common denominator of ten. It is possible, similar to the case of April previously discussed, that Christy reasoned this way because of a limited proficiency with equivalent fractions.

Research Question 1b: Reasoning About the Decimal-Fraction Relationship

This section addresses Research Question 1b regarding the students' reasoning about the relationship between fractions and decimals. This section presents an analysis of the strategies students used to convert between fractions and decimals that takes into consideration the types of strategies students used, as well as how the denominator of the relevant fraction influenced the types of strategies students used during the conversions. In particular, the analysis of students' reasoning during the conversion tasks revealed five findings concerning students' understanding of the relationship between fractions and decimals: (1) Students possessed knowledge of fraction-decimal equivalences for several benchmark quantities. (2) Students made essential use of benchmark knowledge to support conversion reasoning. (3) Students were able to draw on number fact fluency to support conversions between fractions and decimals. (4) Students used number facts and relationships between unit fractions and their decimal equivalents to make conversions. (5) Students used halving, doubling, and disembedding to make conversions. This section describes how the data from the study supported these five findings.

In the following, the researcher reports only students' mathematically correct strategies for converting between fractions and decimals. One reason for this is that in the few cases of mathematically incorrect conversions, students were typically attempting to use a mathematically correct type of strategy that incorporated a computational error. Furthermore, in cases of incorrect conversions, once students used the applets to construct the asked for (but incorrect) equivalent number, visual feedback from the applet would inform students that the resulting number was not equivalent, and they would realize they made an error. It was very common for students to successfully troubleshoot their computations and correct their errors to make correct conversions.

Terminology for Conversion Strategies

The researcher defines a *decimal-fraction conversion strategy* to be an approach or method a student uses for the purpose of converting between a fraction and a decimal, where the approach or method consists of the application of a particular sequence of component mathematical operations, such as multiplication, division, addition, or subtraction of quantities.

During conversion tasks students were asked to convert numbers of a *given number type* (fractions or decimals) to a *target number type* (decimals or fractions). For instance, if a task requests a student to convert 0.2 to a fraction, then the *given number type* is a decimal and the *target number type* is a fraction.

Numbers of Observed Conversion Explanations

During the course of the data collection, the four students produced 274 explanations of conversions between fractions and decimals. Table 3 depicts the number of explanations of conversions that occurred during tasks of converting fractions to decimals, decimals to fractions, and during fraction and decimal comparison tasks.

Table 4 shows the number of conversion explanations offered by each of the four students.

Table 3

Number of Observed Explanations of Conversions between Fractions and Decimals

Explanations by type of task	Number of explanations
Explanations during fraction to decimal tasks	110
Explanations during decimal to fraction tasks	105
Explanations during fraction and decimal comparison tasks	59
Total explanations of conversions	274

Table 4

Number of Conversion Explanations by Student

Student	Number of explanations
April	65
Dan	50
Christy	67
Lisa	92

Finding 1: Benchmark Knowledge of Fraction and Decimal Equivalences

In numerous instances, students referred to and drew on knowledge of the relationship between fractions and decimals for basic benchmark quantities. *Benchmark knowledge* refers to any prior knowledge possessed by a student about the relationship between fractions and decimals for specific, commonly taught quantities. Of the 274 explanations of conversions offered by students, 15 of those explanations were based on *benchmark knowledge*.

The following transcript excerpt from the fourth task-based interview with April illustrates the use of *benchmark knowledge* during the conversion of 6/8 to a decimal:

Interviewer: So, how about let's do the sixth one, so six eighths.
April: Three-fourths, it's point seventy five.

Observe that April uses her understanding of fraction equivalence in the above transcript excerpt to recognize that 6/8 reduces to ¾, where she was then able to draw on her benchmark knowledge to identify 0.75 as the decimal equivalent of ¾. Furthermore, April immediately recognized the numerical relationship between ¾ and 0.75, which indicates her familiarity with this relationship.

Benchmark knowledge during fraction to decimal conversions. During tasks of converting fractions to decimal, the students used their *benchmark knowledge* during 11 of the 110 explanations of the conversion of fractions to decimals. Table 5 below shows how students' use of *benchmark knowledge* varied according to the denominator of the given fraction. As can be seen in the table, each of the four participants made use of benchmark knowledge during fraction to decimal conversions.

Table 5 indicates that students mentioned *benchmark knowledge* most frequently during tasks of converting fractions to decimals when the given fractions contained denominators of eight, where students mentioned *benchmark knowledge* during eight such explanations. A plausible reason for students' greater use of *benchmark knowledge* during these types of conversions versus the other types of conversions is they likely lacked multiplication facts or other number facts they could recall to aid in making the conversions. In a number of instances where students were asked to convert a fraction such as 6/8 to a decimal, they realized the fraction can be reduced to a benchmark fraction, in this case to ¾, use *benchmark knowledge* to reason that ¾ = 0.75, and thus it must be true that 6/8 = 0.75. Hence, students made use of their understanding of fraction equivalence during these instances of reducing fractions with denominators of eight to benchmark decimals.

Table 5

Participants' Use of Benchmark Knowledge for each Denominator Type during Fraction to Decimal Conversions

Denominator	5	20	25	8
April	0	1	0	3
Dan	2	0	0	2
Christy	0	0	0	1
Lisa	0	0	0	2
Frequency count of all participants' use of the *benchmark knowledge* for fraction to decimal conversions	2	1	0	8

Benchmark knowledge during decimal to fraction conversions. Students mentioned *benchmark knowledge* during four explanations of the conversion of decimals to fractions. The following transcript excerpt, from the third task-based interview with Christy, illustrates her use of *benchmark knowledge* during the task of converting 0.75 to a fraction with a denominator of 20:

> Interviewer: Okay, good. So, what's that one? What's that decimal there?
> Christy: Fifteen twentieths, or three fourths, or, um, point seven five.

In the above transcript excerpt, Christy was given the unlabeled decimal quantity 0.75 on a number line marked in tenths, prompted to interpret this decimal quantity, and to determine its equivalent as a fraction with a denominator of 20. Three observations can be made of Christy's explanation: First, Christy identifies 15/20 as the asked for fraction. Second, Christy mentions that 15/20 is equivalent to 3/4. Third, Christy draws on her *benchmark knowledge* by mentioning that 3/4 is equivalent to 0.75. Here, we observe that Christy mentions the asked for fraction 15/20 before mentioning the equivalence 3/4 = 0.75, which highlights the incidental role this *benchmark knowledge* played in Christy's

Table 6

Participants' Use of Benchmark Knowledge for each Denominator Type During Decimal to Fraction Conversions

Denominator	5	20	25	8
April	1	0	0	0
Dan	0	0	0	0
Christy	0	2	0	0
Lisa	0	1	0	0
Frequency count of all participants' use of the *benchmark knowledge* for decimal to fraction conversions	1	3	0	0

conversion reasoning.

There were differences between how students' used *benchmark knowledge* during conversions of decimals to fractions and conversions of fractions to decimals. One difference was that the students' made less use of *benchmark knowledge* during conversions of decimals to fractions than for fractions to decimals. Indeed, as shown below in Table 6, students used benchmark knowledge during only four explanations of the conversion of decimals to fractions, where April, Christy, and Lisa showed evidence of the use of this strategy.

Another difference is that *benchmark knowledge* played a more incidental role in students' explanations of the conversion of decimals to fractions than fractions to decimals, as illustrated in the above transcript of Christy's explanation, where students typically did not use *benchmark knowledge* as a key part of their reasoning for the conversion. Note that students received equivalent opportunities of applying *benchmark knowledge* during fraction to decimal and decimal to fraction conversion tasks since they were presented with equal numbers of these types of tasks.

Finding 2: Strategies based on Benchmark Knowledge

Students made essential use of *benchmark knowledge* during conversions using the strategy of *benchmark and unit*. The researcher coded a conversion strategy in the category of *benchmark and unit* when the explanation included adding (or subtracting) specific amounts of a unit fraction and its decimal equivalent to (or from) a benchmark equivalence. For example, a student might use the *benchmark and unit* strategy to reason that 11/20 converts to 0.55, by reasoning that since 10/20 = 0.5 and 1/20 = 0.05, and that the addition these two equivalences yields the relationship 11/20 = 0.55. The *benchmark and unit* strategy allows students to make conversions between fractions and decimals for quantities close in value to benchmark quantities.

Benchmark and unit strategy during fraction to decimal conversions. Students used the *benchmark and unit strategy* during six explanations of the conversion of fractions to decimals. The following transcript excerpt, from the fourth task-based interview with Dan, illustrates his use of the *benchmark and unit strategy* in his explanation of the conversion of 5/8 to the decimal 0.625.

Interviewer: So, for five eighths, so, that's, five eighths is point six two five. How do you know that? I saw you pretty much just make that without even adjusting the fraction. So, how did you know that?

Dan: Point five plus point one two five, point five plus point one is point six, and we can just leave the point two five be intact onto the end.

We see in the above excerpt that Dan applies the strategy of *benchmark and unit* when he adds 0.125, the decimal equivalent of 1/8, to the benchmark quantity 0.5 to obtain 0.625.

Table 7

Participants' Use of Benchmark and Unit Strategy for each Denominator Type During Fraction to Decimal Conversions

Denominator	5	20	25	8
April	0	0	0	2
Dan	0	0	0	1
Christy	0	0	0	1
Lisa	0	0	0	2
Frequency count of all participants' use of the *benchmark and unit* strategy for fraction to decimal conversions	0	0	0	6

Table 7 shows how students' use of the *benchmark and unit strategy* varied during fraction to decimal conversions according to the denominator of the given fraction. As can be seen in Table 7, each of the four participants showed evidence of use of the benchmark and unit strategy.

Observe in Table 7 that students used the *benchmark and unit strategy* only during conversions of fractions with denominators of eight to decimals. A likely reason for this finding is that students lacked multiplication facts or other number facts they could easily recall to support these conversions, so that the students resorted to other types of strategies, including strategies involving benchmark quantities. The researcher also noted that the fractions for which students applied the *benchmark and unit strategy* included the conversions of 3/8, 5/8, and 7/8 to decimals. Each of these fractions is either between a pair of benchmark quantities or, in the case of 7/8, between a benchmark quantity and 1. The proximity of these fractions to benchmark quantities was a possible factor in the students' use of the *benchmark and unit strategy* during conversions of fractions with denominators of eight to decimals.

Benchmark and unit strategy during decimal to fraction conversions. Students used the *benchmark and unit* strategy during two explanations of decimal to fraction conversions. The following transcript excerpt, from the fourth task-based interview with Christy, illustrates her use of the *benchmark and unit strategy* during an explanation of the conversion of 0.625 to 5/8.

044	Interviewer	So, what fraction that is, point six two five?
045	Christy	Oops, five eighths [Christy uses the applet to make 5/8].
046	Interviewer	Interviewer: Okay, so how does that make sense mathematically? Why do you think...?
047	Christy	Christy: Well....
048	Interviewer	Can you explain that to me?
049	Christy	Well, first of all they match up, and then second they, point six two five is pretty much one ahead of, like, point one two five ahead of half, or four eighths, so, uh huh.

We can see from line 045 that Christy used the applet to make 5/8 as a fraction equivalent to the given decimal 0.625. After additional questioning from the interviewer, in line 049 Christy uses the *benchmark and unit strategy* when explaining that 0.625 is 0.125 more than 0.5 = ½, and using her benchmark knowledge to identify that 0.5 and 4/8 are equivalent.

Table 8 shows the frequency count for the students' use of the *benchmark and unit* strategy for each of the target denominator types, where Christy and Lisa were the only two participants who used this strategy during decimal to fraction conversions.

Table 8

Participants' Use of Benchmark and Unit strategy for each Denominator Type During Decimal to Fraction Conversions

Denominator	5	20	25	8
April	0	0	0	0
Dan	0	0	0	0
Christy	0	0	0	1
Lisa	0	0	0	1
Frequency count of all participants' use of the *benchmark and unit* strategy for decimal to fraction conversions	0	0	0	2

We see in Table 8 above that each of these uses of the *benchmark and unit* strategy was for conversions of decimals to fractions with denominators of eight. As in the case of conversions of fractions to decimals, the students' use of this strategy for conversions of decimals to fractions with denominators of eight is likely the result of their lack of convenient number facts to facilitate such conversions.

Finding 3: Conversions and Number Fact Fluency

Strategies based on scaling up. Students used the conversion strategies of *scaling up* and *reducing* to make conversions between fractions and decimals, where both of these strategies make essential use of students' proficiency with multiplication and division number facts.

Scaling up and students' use of multiplication facts. *Scaling up* is a strategy for conversions between fractions and decimals based on multiplication. The *scaling up* strategy makes essential use of fraction equivalence and involves the multiplication of each of the numerator and denominator by a *scaling factor* that yields an equivalent fraction with a denominator of 10 or 100. The student then recognizes that by convention the resulting equivalent fraction with a denominator of 10 or 100 is equal to a decimal in tenths or hundredths. An example of an application of the *scaling up* strategy is to the conversion of 11/20 to a decimal, where the student applies the *scaling facto*r of 5 to the numerator and denominator of 11/20 to obtain the equivalent fraction of 55/100, and the then recognizes that 55/100 is equivalent to the decimal 0.55. Thus, the researcher coded conversions between fractions and decimals in the category of *scaling up* strategy if the reasoning involved a combination of fraction equivalence and use of a *scaling factor* as the basis of conversion.

Note that, as described above, *scaling up* is a strategy that is readily applicable to the conversion of fractions to decimals. However, as is subsequently described in this section, students did not use the *scaling up* strategy solely for conversions of fractions to decimals, but also for some conversions of decimals to fractions. As a result, *scaling up* was a strategy commonly used by the students, where they used the strategy during 79 explanations of conversions between fractions and decimals.

Scaling up during fraction to decimal conversions. Students used the *scaling up* strategy during 51 explanations of the conversion of fractions to decimals. The following is a transcript excerpt, which occurred during the third task-based interview with April, illustrates her use of the *scaling up* strategy in her explanation of the conversion of the fraction 3/20 to the decimal 0.15:

Interviewer: Point one five. So how does that convert?
April: Five, again five times twenty equals a hundred, and that's like a place value of it. So, if I did the denominator, then I'd need to do the numerator, so that would be fifteen, so fifteen hundredths, or in

> decimal form [gestures at her construction of 0.15 on the computer screen].

We see in this transcript excerpt that April is using five as a *scaling factor* when she mentions, “five times twenty equals a hundred.” Additionally, she refers to multiplying the numerator of 3/20 by the *scaling factor* of five when she mentions, “I’d need to do the numerator,” and explains that the result “would be fifteen, so fifteen hundredths”.

Table 9 shows the frequency count of students’ use of the *scaling up* strategy for each of the given denominator types. It is evident from Table 9 that each of the four participants made use of this strategy during conversions of fractions to decimals. Table 9 indicates that students’ primary use the *scaling up* strategy was for conversion tasks of given fractions with denominators of 20 or 25. Students applied the *scaling up* strategy in a straightforward manner, as indicated above, and described using a *scaling factor* to scale up the numerator and denominator of the given fraction to convert the given fraction to a fraction over the denominator of 10 or 100. Students’ knowledge

Table 9

Participants’ Use of Scaling Up for each Denominator Type during Fraction to Decimal Conversions

Denominator	5	20	25	8
April	3	5	6	0
Dan	0	5	4	0
Christy	0	4	10	0
Lisa	1	6	7	0
Frequency count of participants’ use of the *scaling up* strategy by denominator type during fraction to decimal conversions	4	20	27	0

of multiplication number facts they were able to recall played a strong role in supporting their use of the *scaling up* strategy for conversions of fractions to decimals.

To convert a given fraction with a denominator of 20 (such as 9/20) to a decimal by *scaling up*, students used a *scaling factor* of five to obtain an equivalent fraction with a denominator of 100. Similarly, to convert a given fraction with a denominator of 25 (such as 17/25) to a decimal using the *scaling up* strategy, students used the *scaling factor* of four to obtain an equivalent fraction with a denominator of 100.

The reader will note that Table 9 above indicates that no students successfully used the *scaling up* strategy to convert fractions to decimals when the denominator was eight. The students likely did not have any easily recallable multiplication facts that would allow them to apply the *scaling up* strategy to convert a fraction such as 3/8 to the equivalent fraction of 375/1000 using the *scaling factor* of 125. Since students lacked the multiplication number facts necessary to successfully apply the *scaling up* strategy to fractions with denominators of eight, and possibly chose other strategies to make fraction to decimal conversions in these cases.

Scaling up during decimal to fraction conversions. *Scaling up* was a commonly used strategy for decimal to fraction conversions, where the students used *scaling up* during 28 explanations of the conversion of decimals to fractions.

Below is an excerpt of a Christy's explanation involving the *scaling up* strategy using the *scaling factor* of 5 in her explanation of the conversion of 0.85 to 17/20. This excerpt was taken from the third task-based session with Christy.

Interviewer: Okay, so, is there some way of understanding that these are actually equal?

Christy: Um...um, there's also another way in the twentieths, and then I can do seventeen times five, and then that will equal, I'm pretty sure that that will equal eighty-five. And then doing...yeah, eighty-five.

We can see in Christy's explanation that she anticipated that 17/20 *scales up* to 85/100 using the *scaling factor* of five, by describing 85 as being the product of 17 and 5.

The analysis of the transcript data revealed that the target denominator type influenced students' use of the *scaling up* strategy during conversions of decimals to fractions. Table 10 shows the frequency counts of students' use of the *scaling up* strategy for the different types of denominators of the *target* fractions, where it is evident that each of the four students made use of this strategy.

Similar to the case of conversions of fractions to decimals, students primarily used of the *scaling up* strategy for conversions of decimals to fractions that involved *target* fractions with denominators of 20 or 25. Students used the *scaling factor* of four for conversions to fractions with denominators of 25, and the *scaling factor* of five for conversions to fractions with denominators of 20. Note also that no students used the

Table 10

Participants' Use of Scaling Up for each Denominator Type during Decimal to Fraction Conversions

Denominator	5	20	25	8
April	0	4	1	0
Dan	1	1	2	0
Christy	4	7	1	0
Lisa	0	1	6	0
Frequency count of all participants' use of the *scaling up* strategy for different denominators during decimal to fraction conversions	5	13	10	0

scaling up strategies for conversions of decimals to fractions for fractions with denominators of eight, since these conversions involve the large and unwieldy *scaling factor* of 125.

Reducing and students' use of division facts. *Reducing* is a strategy for conversions between fractions and decimals based on division, and the researcher coded conversion strategies in the category of *reducing* when conversions reasoning involved removing common factors of a fraction by dividing the numerator and denominator by a common *reducing factor*.

Reducing during fraction to decimal conversions. Students used *reducing* during 12 explanations of conversions of fractions to decimals. Below is an example of Lisa's use of *reducing* to convert 4/5 to the decimal 0.8, which occurred during the initial clinical interview with Lisa.

Interviewer: Okay, so let's try this, instead of decimals we had fractions. Okay, so let's say we had that one. What would the decimal be for that? So, now it's make a decimal.

Lisa: Okay, so...it's point eight, because eight tenths, because eight tenths, you know, it's eight tenths. And then eight over ten, and then we can change that to make it divided by two, it's four, and then ten divided by two is five, so it's four fifths.

We can see that Lisa applies the *reducing* strategy in this conversion, because she mentions dividing both the numerator and denominator of 8/10 by the *reducing factor* of 2, which results in the reduced fraction of 4/5.

A variation of students' use of *reducing* during fraction to decimal conversions occurred when students reduced a given fraction to a benchmark quantity, and then used the fraction-decimal relationship for the benchmark quantity. Below is an excerpt illustrating this type of use of the *reducing* strategy by Christy in her explanation of the conversion of 5/20 to 0.25, which occurred during the third task-based interview with Christy.

> Interviewer: So, how did you know that?
> Christy: Because, I simplified that [indicates 5/20 on the computer screen] and then it's one-fourth, so, and then one-fourth in decimal form is point two five.

We can see in the above excerpt that Christy recognized that 5/20 reduces to ¼, and subsequently recognized that ¼ is equivalent to 0.25.

Table 11 shows the frequency count of students' use of *reducing* for fraction to decimal conversions for the different denominator types of the given fractions. Table 11 indicates that each of the four students made use of the reducing strategy during fraction to decimal conversions.

We can see from Table 11 that a few more students used *reducing* during fraction to decimal conversions involving reducing fractions containing common factors in the numerator and denominator. A few more students used denominators of 20 as a

Table 11

Participants' Use of Reducing for each Denominator Type during Fraction to Decimal Conversions

Denominator	5	20	25	8
April	0	0	0	3
Dan	0	0	1	0
Christy	0	1	2	0
Lisa	1	4	0	0
Frequency count of participants' use of the *reducing* strategy by denominator type during fraction to decimal conversions	1	5	3	3

reducing strategy during fraction to decimal conversions. In these cases, students made use of the *reducing factor* of two during conversions by dividing the numerator and denominator of the given fraction by the common factor of two.

Reducing during decimal to fraction conversions. Participants made greater use of the *reducing* strategy during decimal to fraction conversions than for fraction to decimal conversions. In particular, students used *reducing* during 37 explanations of the conversion of decimals to fractions.

The typical use of *reducing* during these types of tasks was to convert the given decimal to a fraction with a denominator of 100 or 10, and subsequently to divide the numerator and denominator of the fraction by a suitable *reducing factor* to reduce the fraction. The following excerpt, which occurred during the third task-based interview with Dan, illustrates his use of this strategy during his explanation of the conversion of 0.72 to 18/25.

> Interviewer: So, what's that going to be as a fraction?
> Dan: Seventy-two divided by four, well, I know seventy-two divided by eight is nine, which would be eighteen twenty-fifths.

In the above transcript excerpt we see that Dan applied the *reducing* strategy by computing the quotient of 72 divided by 4 by doubling the result of 72 divided by 8, which gives him 18. He then mentions that the resulting conversion is 18/25.

During conversions of decimals to fractions, the *target* fraction denominator influenced students' use of *reducing* as a strategy. Table 12 shows how the frequency count of explanations based on *reducing* varied according to the *target* denominator type. It is evident from Table 12 that each of the four students used the reducing strategy

Table 12

Participants' Use of Reducing for each Denominator Type during Decimal to Fraction Conversions

Denominator	5	20	25	8
April	4	1	3	0
Dan	0	2	4	0
Christy	1	1	5	0
Lisa	3	9	5	0
Frequency count of all participants' use of the *reducing* strategy for different denominators during decimal to fraction conversions	8	13	17	0

during conversions of decimals to fractions.

The participants' use of the *reducing* strategy, particularly for converting decimals to fractions with denominators of 20 and 25, reflects their knowledge and proficiency with division number facts. Indeed, the researcher noted that no students used a *reducing* strategy in their explanations of the conversion of decimals to fractions with denominators of 8. A likely explanation for students' lack of use of the *reducing* strategy for denominators of 8 is their lack of knowledge of convenient number facts allowing them to reduce fractions such as 125/1000, 375/1000, 625/1000 and 875/1000 to fractions with denominators of 8. These findings suggest that proficiency with multiplication and division number facts can contribute to and support students' conversions between fractions and decimals.

Finding 4: Using Number Facts and Relationships between Unit Fractions and Decimals

Students used three strategies that incorporated both number fact fluency and unit fraction-decimal relationships in their explanations of conversions between fractions and decimals. These three strategies were *multiplication of units*, *scaling up and adding or subtracting units*, and *addition of units from a base unit*. This section describes how students used these three strategies to make conversions based on number fact fluency and reasoning about the relationship between fractions and decimals.

Multiplication of units. The *multiplication of units* strategy was the strategy that the students used most commonly for fraction and decimal conversions that involved operations with a unit fraction and its decimal equivalent. The researcher coded a conversion strategy in the category of *multiplication of units* strategy when the conversion was accomplished by the simultaneous multiplication of a unit fraction and its equivalent decimal by a whole number. This strategy is similar to the previously described *scaling up* strategy; however, as students applied the *multiplication of units* strategy, their descriptions included the role of unit quantities as an essential part of their conversion reasoning.

Multiplication of units during conversions of fractions to decimals. Participants used the *multiplication of units* strategy during 10 explanations of conversions of fractions to decimals. The following transcript excerpt, taken from the initial clinical interview with Christy, exemplifies her use of the *multiplication of units strategy* during her explanation of how 3/5 converts to 0.6.

Interviewer: Okay, so how do you know for sure, again, that point six is the same as three-fifths?

Christy: Um, it's the same as three-fifths, because, then again it's, uh, the five, the denominator of three fifths, and then that times point two, no, sorry, and then the numerator three times point two and then it's point six.

We observe that in Christy's explanation she mentions that the product of 0.2 and 3 is 0.6, which forms the basis for her reasoning that 3/5 = 0.6, and where she is implicitly using the fact that 1/5 = 0.2. One interpretation is that when explaining Christy may have drawn on her understanding of rules for multiplying decimals in her explanation of this conversion.

Participants used the *multiplication of units* strategy during 10 explanations of the conversion of fractions to decimals. The type of denominator of given fractions appeared to influence students' use of this strategy during conversions of fractions to decimals, where students used this strategy least for conversions of fractions with denominators of 5 to decimals. Table 13 below depicts the relationship between the frequency counts of participants' use of the *multiplication of units* strategy and the type of denominator. The multiplication of units strategy was only used by April and Lisa during conversions of fractions to decimals.

Students again drew on their knowledge of multiplication facts during conversions involving fractions with denominators of 20 and 25, which explains their use of *multiplication of units* strategy during these tasks.

Table 13

Participants' Use of Multiplication of Units for each Denominator Type during Fraction to Decimal Conversions

Denominator	5	20	25	8
April	0	2	1	2
Dan	0	0	0	0
Christy	1	0	0	0
Lisa	0	1	3	1
Frequency count of all participants' use of the *multiplication of units* strategy for different denominators during fraction to decimal conversions	1	3	4	3

Multiplication of units during conversions of decimals to fractions. Students April, Lisa, and Christy used the *multiplication of units* strategy during 17 explanations of conversions of fractions to decimals. A common feature of these explanations is students' conversion of a given fraction to a decimal by multiplying the decimal equivalent of the unit fraction by the numerator of the given fraction. An interpretation of this common use of the *multiplication of units* strategy is that place value rules for the multiplication of decimals by whole numbers played a fundamental role in their understanding and explanation of these conversions, because they performed these multiplication operations on decimals, the *target number type* of the conversions.

Students also used the *multiplication of units* strategy during explanations of the conversion of decimals to fractions. The following transcript excerpt from the third task-based interview with April illustrates her use of this strategy during the decimal to fraction conversion of 0.15 to 3/20:

Interviewer: So, given that point Oh five is one twentieth, what would be the fraction for point one five? What do you think that would be?

April: Well, that's fifteen hundredths, so... I'll just go back, okay, three twentieths. Three twentieths, because, again the five, the point Oh five is five times twenty, and since five times three equals fifteen, which is the decimal, that would be three over twenty, because it's three times, to get, like the fraction, the decimal is five times three, so, since last time it was one twentieth, it would be three twentieths.

In the above transcript excerpt, the interviewer specifically prompts April concerning the relationship between 0.05 and 1/20. In response, when April mentions “five times three” she appears to indicate multiplying 0.05 by 3 to obtain 0.15. In addition, she mentions multiplying 3 by 1/20 to obtain the resulting fraction of 3/20.

The *target* denominator type appeared to influence the frequency of students’ use of the *multiplication of units* strategy, where this strategy was used especially frequently when the target denominator was 20. Table 14 shows the frequency count of the *multiplication of units* strategy for each *target* denominator type, where April, Christy, and Lisa evidenced use of this strategy during conversions of decimals to fractions. Students drew on their knowledge of multiplication number facts during explanations

Table 14

Participants’ Use of Multiplication of Units for each Denominator Type during Decimal to Fraction Conversions

Denominator	5	20	25	8
April	0	5	2	2
Dan	0	0	0	0
Christy	1	1	0	0
Lisa	0	4	0	1
Frequency count of all participants’ use of the *multiplication of units* strategy for different denominators during decimal to fraction conversions	1	10	2	3

based on this strategy, which appears to explain the large number of conversions involving fractions with denominators of 20. In particular, the students' easy recollection of multiples of five appeared to facilitate their use of this strategy for decimal to fraction conversions involving fractions with denominators of 20, which contributed to the increased frequency count of conversions involving this particular denominator.

Addition of units from a base unit. The *addition of units from a base unit* strategy includes explanations based on the repeated addition of the quantities in a unit fraction-decimal relationship. Thus, the researcher coded a strategy in the category of *addition of units from a base unit* if the conversion reasoning involved the repeated addition of a unit fraction and its decimal equivalent.

Christy and Lisa were the only students who used the *addition of units from a base unit* strategy for conversion of decimals to fractions, and no students used this strategy for conversions of fractions to decimals. The following transcript excerpt, from the fourth task-based interview with Lisa, exemplifies her use of this strategy during an explanation of the conversion of 0.375 to 3/8:

050	Interviewer:	So, now it's giving you a decimal, and asking…okay…what is that fraction that corresponds to that one?
051	Lisa	Okay, so, thirty-seven. Ah, what was I doing? Yeah, okay, and then find the fraction, okay.
052	Interviewer	What do you think that will be?
053	Lisa	So, they're eighths, yeah, and then (laughs), three seven five.
054	Interviewer	Well, for one thing, you see the point right here, right? [Indicates a point on the computer screen]
055	Lisa	Uh huh.
056	Interviewer	So, does that seem to line up with one of those?
057	Lisa	Yeah...three eighths, but, okay, so it's one two five plus one two five is twenty -five, plus one two five is, yeah.

Table 15

Participants' Use of Addition of Units from a Base Unit for each Denominator Type during Decimal to Fraction Conversions

Denominator	5	20	25	8
April	0	0	0	0
Dan	0	0	0	0
Christy	1	1	0	0
Lisa	0	0	0	2
Frequency count of all participants' use of the *addition of units from a base unit* strategy for different denominators during decimal to fraction conversions	1	1	0	2

Since the equivalence of 0.125 = 1/8 is a relationship given for this task, the researcher interpreted Lisa's reasoning in line 057 as an attempt to understand 3/8 as the quantity 0.125 added to itself three times.

Participating students used the *addition of units from a base unit* strategy during four explanations of conversions of decimals to fractions. Table 15 shows the frequency count of the students' use of the *addition of units from a base unit* strategy for each of the four *target* fraction denominator types, where Christy and Lisa were the only students who made use of this strategy during conversions of decimals to fractions.

Participants used this strategy to add the decimal equivalent of the base unit fraction, as illustrated by Lisa did in the above transcript excerpt, where she added 0.125, the decimal equivalent of 1/8, to itself. It is possible that students chose to add these decimals because their place-value based understanding of the addition of decimals made it feasible for them to understand and phrase their explanations in terms of the repeated addition of a decimal quantity. This perhaps also explains why the students did not offer explanations based on the *addition of units from a base unit* strategy during conversions of fractions to decimals.

Scaling up and adding or subtracting units. The researcher coded a conversion strategy in the category of *scaling up and adding or subtracting units* when conversion reasoning involved the use of an appropriate *scaling factor* to scale a fraction up to its equivalent (in hundredths), with the addition or subtraction of specific amounts of the unit fraction and its equivalent decimal, to obtain the relevant conversion. For instance, in the case of the conversion of 19/25 to a decimal, a student might reason that 20/25 must be equivalent to the decimal 0.80 by using the *scaling factor* of 4, and then subtract 1/25 = 0.04 from each side of the equivalence of 20/25 = 0.80 to conclude that 19/25 converts to 0.76. Students made use of multiplication number facts while using this strategy, based on their use of *scaling factors* during explanations of conversions.

Scaling up and adding or subtracting units during fraction to decimal conversions. Students used the *scaling up and adding or subtracting units* strategy during six explanations of conversions of fractions to decimals. The following transcript excerpt, which occurred during the third task-based interview with Lisa, illustrates her use of this strategy during the conversion of 14/20 to 0.7:

> Interviewer: Okay, so what about fourteen [twentieths]?
> Lisa: Okay, so, twelve times five is sixty, plus ten is seventy.

In the above transcript excerpt, we can interpret Lisa as having reasoned that 12/20 converts to 0.60 using the *scaling factor* of 5, to obtain the equivalence of 14/20 = 0.70 by adding two units of 1/20 = 0.05 to each side of 12/20 = 0.60.

Table 16

Participants' Use of Scaling Up and Adding or Subtracting Units for each Denominator Type during Fraction to Decimal Conversions

Denominator	5	20	25	8
April	0	0	0	0
Dan	0	0	3	0
Christy	0	0	0	0
Lisa	0	3	0	0
Frequency count of all participants' use of the *scaling up and adding or subtracting units* strategy for different denominators during fraction to decimal conversions	0	3	3	0

Table 16 above shows the frequency counts of students' use of this strategy for the four different denominator types of given fractions, where Dan and Lisa were the only two students who used this strategy during conversions of fractions to decimals.

A likely reason for students' use of this strategy during conversions for fractions with denominators of 20 and 25 is that this strategy makes essential use of multiplication. This is because the students knew many multiplication facts involving the *scaling factors* of five and four, which likely facilitated their use of this strategy during conversions involving fractions with denominators of 20 and 25.

Scaling up and adding or subtracting units during decimal to fraction conversions. Students made use of the *scaling up and adding or subtracting units* strategy during four explanations of conversions of decimals to fractions. The following transcript excerpt, taken from the third task-based interview with Lisa, illustrates her use of this strategy during her explanation of the conversion of 0.76 to 19/25.

Interviewer: So, point seven six. What do you think that will be?

Lisa: Okay, so, I know that it, four times fifteen is sixty, and then four times sixteen is sixty-four, and then four times seventeen is...sixty eight, and then four times eighteen is seventy-two, so four times nineteen (laughs). And then this is seventy-six.

We can see from Lisa's reasoning that she evidently applied this strategy since she started with the relationship of 15/25, obtained by using the *scaling factor* of four, and then added four units of 1/25 = 0.04 to build up to the relationship of 0.76 = 19/25.

Table 17 shows the frequency counts of students' use of this strategy during conversions of decimals to fractions for each of the four *target* denominator types, where Dan and Lisa were the only two students who used this strategy during conversions of decimals to fractions.

Scaling up and adding or subtracting units is not an easy strategy to apply in the sense that is requires a student to simultaneously keep track of how many multiples of a unit fraction and its decimal equivalent are added to or subtracted from a base equivalence that itself is obtained using the *scaling up* strategy. These requirements

Table 17

Participants' Use of Scaling Up and Adding or Subtracting Units for each Denominator Type during Decimal to Fraction Conversions

Denominator	5	20	25	8
April	0	0	0	0
Dan	0	0	1	0
Christy	0	0	0	0
Lisa	0	1	2	0
Frequency count of all participants' use of the *scaling up and adding or subtracting units* strategy for different denominators during decimal to fraction conversions	0	1	3	0

perhaps explain students' infrequent use of the strategy during conversion tasks.

Finding 5: Three Other Conversion Strategies

Students used three additional strategies during conversions between fractions and decimals, including *halving*, *doubling*, and *disembedding*. *Halving* and *doubling* in particular are strategies other researchers have observed students using during conversions between fractions and decimals (Moss & Case, 1999). This section describes students' use of these three strategies during conversions between fractions and decimals.

Halving during conversions. *Halving* is a conversion strategy in which a student knows a fraction-decimal equivalence for a particular quantity, such as in the case of *benchmark knowledge*. The student then deduces a new fraction-decimal equivalence by taking half of both the fraction and decimal of the known equivalence. The researcher coded conversion explanations in the category of *halving* that included the above characteristics. *Halving* was the least used conversion strategy, where students used *halving* during four explanations of conversions between fractions and decimals.

The following transcript excerpt, which occurred during the fourth task-based interview with Dan, illustrates his use of *halving* during his explanation of the conversion of 1/8 to 0.125.

058	Interviewer	Yeah, so this is eighths. So, what is the decimal that is equivalent to one eighth?
059	Dan	Point one two five.
060	Interviewer	Point one two five. Did you know that already?
061	Dan	Uh huh.
062	Interviewer	You knew that, okay, alright. So, how did you know that, anyway? Is that something you learned from the math lessons?
063	Dan	Well, two eighths is two fourths, so point two five is

> two eighths. Point two five divided by two is point one two five.

It is apparent from the line 063 that Dan reasoned that 1/8 converts to 0.125 because 1/4 is equivalent to 0.25, that 1/4 = 2/8, that it must be true that the decimal for 1/8 must be half of 0.25, which is 0.125. Table 18 shows the frequency counts of students' use of the *halving* strategy during conversions of fractions to decimals.

Table 18 indicates that students only used *halving* for fraction to decimal conversion tasks where the given denominator type was eight. In their explanations of the conversion of 1/8 to 0.125, Dan, Christy, and Lisa each used the *halving* strategy (where Dan's explanation is in the preceding transcript excerpt). A plausible explanation for these students' consistent use of *halving* during this task is they lacked arithmetic number facts that would allow them to convert 1/8 to 0.125, and used *halving* as a viable alternative strategy.

Table 18

Participants' Use of Halving for each Denominator Type during Fraction to Decimal Conversions

Denominator	5	20	25	8
April	0	0	0	0
Dan	0	0	0	1
Christy	0	0	0	1
Lisa	0	0	0	1
Frequency count of all participants' use of *halving* for different denominators during fraction to decimal conversions	0	0	0	3

Christy was the only student who used *halving* during a single decimal to fraction conversion explanation, when using *halving* to explain the conversion of 0.05 to 1/20. The following transcript excerpt occurred during the third task-based interview with Christy and illustrates her reasoning regarding the conversion:

064	Interviewer	Alright, so, for this one, given that point one is one tenth, what do you think the fraction for point Oh five would be. Kind of what is the relationship between point one and point Oh five?
065	Christy	Um, I'm pretty sure that because, I'm pretty sure that point Oh five is going to be one twentieth, because one tenth is also two twentieths. And then the decimal is point five, and then, and then point one is also poi-, one tenth, so I just half one tenth which is one twentieth.
066	Interviewer	One twentieth, okay, so what about, like, the relationship between these two, specifically, point Oh five and point one?
067	Christy	Um, point Oh five is half of point one.

In line 067 of the above excerpt we can see that Christy applies the *halving* strategy, since she mentions the equivalence of 0.1 and 1/10 as well as the equivalence of 1/10 and 2/20, and that 0.05 is half of 0.1. Furthermore, she concludes that 0.05 is equivalent to 1/20 because 0.05 is half of 0.1 and that 1/20 is half of 1/10.

One possible explanation for Christy's use of *halving* as described above is that the task specifically prompted her to find the fraction for 0.05, where the equivalence of 0.1 = 1/10 was given, and it is likely that Christy realized that 0.05 is half of 0.1.

Doubling during conversions. In the category of *doubling* the researcher coded those fraction-decimal conversions that students accomplished by doubling the quantities of another fraction-decimal equivalence, such as from *benchmark knowledge*.

Table 19

Participants' Use of Doubling for each Denominator Type during Fraction to Decimal Conversions

Denominator	5	20	25	8
April	0	0	0	1
Dan	0	0	1	0
Christy	0	0	0	1
Lisa	0	0	0	0
Frequency count of all participants' use of *doubling* for different denominators during fraction to decimal conversions	0	0	1	2

Doubling during conversions of fractions to decimals. Students used *doubling* during three conversions of fractions to decimals. The following transcript excerpt, which occurred during the fourth task-based interview with April, illustrates her use of *doubling* during the task of converting 2/8 to 0.25.

Interviewer: Yeah, so that's what this is, given that one eighth is point one two five, what is the decimal for two eighths?
April: Wouldn't you just double it?

In the above excerpt, when April says "Wouldn't you just double it?" she is apparently referring to doubling the quantities mentioned by the interviewer.

Table 19 shows the frequency counts of students' use of *doubling* during fraction to decimal conversion tasks for the four denominator types of given fractions, and shows that doubling was used by April, Dan, and Christy during conversions of fractions to decimals.

Students may have applied *doubling* during the two conversions that involved denominators of eight because they lacked number facts that might have supported their use of the other previously discussed strategies, and the students likely resorted to alternative strategies such as *doubling* for conversions involving these types of fractions.

Doubling during conversions of decimals to fractions. Dan and Christy each used *doubling* once during conversions of decimals to fractions. The following transcript excerpt, which occurred during the second task-based interview with Christy, illustrates her use of *doubling* during her explanation of the conversion of 0.4 to 2/5:

068	Interviewer	Alright, great, one fifth, alright, very good. So, let's go to the next one. So, given that point two is one fifth, find the fraction for point four. So, again you would make point four, and then find the fraction for that. So, what are you going to do there?
069	Christy	[Uses the applet to make both 0.4 and 2/5]
070	Interviewer	Just two-fifths?
071	Christy	Yeah.
072	Interviewer	Okay.
073	Christy	Because, since point two is one fifth, and then it says give the fraction, oh no, find the fraction for point four. And then I already know that one fifth is point two, so then it says point four so then I just double that, and then now it's two fifths.

Note that in line 073 of the above excerpt that Christy mentions doubling 0.2 to obtain 0.4, and she mentions that 2/5 is equivalent to the given fraction 0.4.

Table 20 shows that Dan and Christy were the only two students who used doubling during conversions of fractions to decimals.

Table 20

Participants' Use of Doubling for each Denominator Type during Decimal to Fraction Conversions

Denominator	5	20	25	8
April	0	0	0	0
Dan	0	1	0	0
Christy	1	0	0	0
Lisa	0	0	0	0
Frequency count of all participants' use of *doubling* for different denominators during decimal to fraction conversions	1	1	0	0

Disembedding. *Disembedding* is a conversion strategy based on the number of unit fractions missing from the whole for a given quantity. Thus, the researcher coded a conversion strategy in the category of *disembedding* if the explanation mentioned the number of unit fractions or the decimal equivalent that are missing from the whole (or one) for the given quantity.

Analysis of the transcript data indicates that *disembedding* strategy was not a commonly used conversion strategy, where participants used the *disembedding* strategy during seven explanations for both conversions of fractions to decimals and decimals to fractions.

Disembedding during fraction to decimal conversions. Students used *disembedding* during two explanations of the conversion of fractions to decimals. The following transcript excerpt, taken from the second clinical interview with Lisa, illustrates her use of this strategy during her explanation of the conversion of 7/8 to 0.875:

074	Interviewer	So, okay, how about for seven eighths?
075	Lisa	Seven eighths? Okay. [Lisa makes 0.875 as the decimal equivalent to 7/8.]
076	Interviewer	Point eight seven five. And you know that because...?
077	Lisa	Because, so I just, I knew it was eight hundred and something, 'cause I remembered. But, I could do the, so, the seven eighths, so I needed one more eighth. So, I needed to get to a thousand. So, I could just do a thousand minus one twenty five is point eight seven five.

The researcher points out that in line 077 Lisa mentions needing to add 1/8 to 7/8, and that in terms of decimals that, thinking in terms of thousandths, she needed to subtract 125 from 1000 to obtain 875. Subsequently, in line 079, Lisa clarifies herself by expressing the need to make the conversion by subtracting 0.125 from one.

Table 21 shows the frequency count for the students' use of *disembedding* for given fractions of each denominator type, where only Christy and Lisa used this strategy during conversions of fractions to decimals.

Table 21

Participants' Use of Disembedding for each Denominator Type during Fraction to Decimal Conversions

Denominator	5	20	25	8
April	0	0	0	0
Dan	0	0	0	0
Christy	0	0	0	1
Lisa	0	0	0	2
Frequency count of students' use of the *disembedding* strategy for different denominators during fraction to decimal conversions	0	0	0	3

Table 21 indicates that students only used *disembedding* during tasks involving given fractions with denominators of eight. A plausible explanation for this finding is that students lacked number facts they could recall allowing them to convert fractions with denominators eight to decimals. Each of the above explanations occurred during the task of converting 7/8 to 0.875.

Disembedding during decimal to fraction conversions. Students also used *disembedding* during five explanations of decimal to fraction conversions. The following transcript excerpt, taken from the third task-based interview with Christy, illustrates her explanation of this strategy during the conversion of 0.95 to 19/20:

080	Interviewer	Point nine five, what do you think that will be as a fraction?
081	Christy	Um, nineteen twentieths.
082	Interviewer	Nineteen twentieths.
083	Christy	Yeah, nineteen twentieths.
084	Interviewer	So, how did you know that? Were you just able to read that off, or...?
085	Christy	Well, there's also a reason, because...well, first of all these two line up. And then also, um, point nine five is point zero five away from being a whole number.

In line 085 of the above excerpt, Christy mentions that 0.95 is 0.05 away from “a whole number,” for which she apparently means one. She appears to implicitly use the fact that 0.05 is equivalent to 1/20, and thus 0.95 is 1/20 less than the whole of 1, and thus 0.95 is equivalent to 19/20.

Table 22 displays the frequency counts of students’ use of the *disembedding* strategy during decimal to fraction conversions for each denominator type. As was the case for conversions of fractions to decimals, Christy and Lisa were the only students who evidenced use of the disembedding strategy during decimal to fraction conversions.

Table 22

Participants' Use of Disembedding for each Denominator Type during Decimal to Fraction Conversions

Denominator	5	20	25	8
April	0	0	0	0
Dan	0	0	0	0
Christy	0	2	1	0
Lisa	0	2	0	1
Frequency count of students' use of the *disembedding* strategy for different denominators during decimal to fraction conversions	0	4	1	1

Students did not initially choose to use this strategy during conversions between fractions and decimals, which is reflected in their infrequent use of this strategy. Indeed, in a number of cases students used this strategy only after the researcher asked the students if there were any other types of conversion reasoning in situations where the given number type is close in value to the number 1. A possible interpretation of this finding is that the students' classroom teacher may not have taught this strategy during their regular classroom instruction.

Research Question 2: Affordances of the Virtual Manipulatives

This section addresses research question 2 regarding the affordances of the virtual manipulatives. This section describes results of the analysis of the evidence of features of the applets that afforded learning opportunities which were revealed in the form of students' hand gestures, mouse cursor motions, and verbal explanations.

Features within the apps afforded opportunities for students to make observations about alignment and partitioning of the quantities. The features that afforded students' recognition of *alignment* and *partition* emerged as the researcher coded and analyzed video and transcript data. *Alignment* and *partition* are examples that would fall under the *efficient precision* category of affordances of virtual manipulatives identified by Moyer-Packenham and Westenskow (2013). In this study, in the context of parallel number lines with the same scaling, *alignment* refers to the fact that two equivalent quantities depicted on parallel number lines will have the same location on the number lines, and will thus appear to be aligned on the parallel number lines. The alignment *affordance* is an example of the *efficient precision* category of affordances of virtual manipulatives, because the applets efficiently and precisely depict the alignment of equivalent quantities on parallel number lines.

The *partition affordance* of the applets belongs to the category of affordances of *efficient precision*, because the applets efficiently and precisely represent fraction and decimal quantities on the parallel number lines, and efficiently and precisely allow students to manipulate fraction and decimal quantities. In particular, while manipulating the applets' denominator slider, students are able to observe how changing the denominator changes the partition on the interval from 0 to 1. Here *partition* refers to a set of points that divide the number line representation of the interval from 0 to 1 into subintervals of equal length.

This section is organized into two subsections. The first subsection provides a description of the evidence of students' awareness of the alignment affordance of applets in the form of students' verbal explanations, hand gestures, and mouse cursor motions. The second subsection describes the evidence of students' awareness of the partition affordance based on the students' verbal explanations, hand gestures and mouse cursor motions. Because each of these four types of applets presents fraction and decimal conversion and comparison tasks to students in different ways, the applets appeared to influence differently students' gestures, mouse behavior, and explanations while using the applets.

Alignment Affordance

Hand gestures indicating alignment. From the video recorded data, the researcher coded a student's hand gesture as indicating *alignment* of points on the two number lines if the student made an up and down motion using either their index finger or their hand. Table 23 shows the frequency counts of each participants' alignment-related hand gestures for each of the three types of conversions. Table 23 indicates that each of

Table 23

Participant Frequency Counts of Alignment-related Hand Gestures by Applet Type

	Dual Construction	One-Way Labeled	One-Way Unlabeled	Total
April	5	6	3	14
Dan	1	1	0	2
Christy	0	10	0	10
Lisa	0	0	2	2
Total	6	17	5	28

Table 24

Participant Frequency Counts of Alignment-related Hand Gestures by Conversion Type

	April	Dan	Christy	Lisa	Total
Decimal to Fraction	8	2	10	2	22
Fraction to Decimal	6	0	0	0	6
Total	14	2	10	2	28

the four participants produced hand gestures indicating alignment while using the conversion applets.

Table 24 shows the frequency count of participants' alignment-related hand gestures according to the type of conversion, decimal to fraction versus fraction to decimal. An interesting feature of Table 24 is that there were 6 alignment-related hand gestures observed during fraction to decimal conversions, whereas there were 22 gestures observed during decimal to fraction conversions. Moreover, the 6 alignment-related gestures observed during fraction to decimal conversions were attributed to April.

Table 25 below shows the frequency count of the four students' hand gestures made while using the *dual construction*, *one-way labeled*, and *one-way unlabeled applets*. Furthermore, Table 25 shows the frequency count of alignment-related hand gestures for decimal to fraction and fraction to decimal conversions.

Table 25 indicates that students produced 22 hand gestures indicating the alignment of points during decimal to fraction tasks, whereas the students produced 6 such gestures during fraction to decimal conversion tasks. One possible interpretation of this difference is that students drew on the *alignment* affordance of the applets more often

Table 25

Frequency Counts of Alignment-related Hand Gestures for each Type of Applet

	Dual Construction	One-way Labeled	One-way Unlabeled	Total Gestures
Decimal to Fraction	5	14	3	22
Fraction to Decimal	1	3	2	6
Total	6	17	5	28

during decimal to fraction conversions than during fraction to decimal conversions. Also note that the largest number of hand gestures indicating alignment occurred while the students used the one-way labeled applets, and that the fewest number of these gestures occurred while students were using the one-way unlabeled applets.

The researcher also coded six *alignment*-related hand gestures as students performed tasks while using the *comparison* applet, three that were attributed to Christy and three that were attributed to Lisa. An explanation of this finding is that students were using alignment-related gestures during comparison tasks in order to compare two given quantities, by using this type of gesture to indicate that two displayed points do not line up and thus one number is greater than the other.

Mouse cursor motions indicating alignment. From the screen captures of students' use of the applets, the researcher coded a movement of the mouse cursor as a *mouse cursor motion indicating alignment* if the cursor motion indicated the vertical alignment of two points on the two number lines. Table 26 shows the frequency count of alignment-related mouse cursor motions for each of the four participants as they used the

Table 26

Participant Frequency Counts of Alignment-related Mouse Cursor Motions by Applet Type

	Dual Construction	One-Way Labeled	One-Way Unlabeled	Total
April	3	4	1	8
Dan	1	0	0	1
Christy	3	8	1	12
Lisa	0	0	0	0
Total	7	12	2	21

three types of conversion applets. Table 26 indicates that April, Dan, and Christy produced mouse cursor motions indicating alignment as they used the applets, whereas Lisa was not observed producing these types of mouse cursor motions.

Table 27 shows the frequency count of alignment-related mouse cursor motions produced by each of the four participants during decimal to fraction conversions versus fraction to decimal conversions.

Table 27

Participant Frequency Counts of Alignment-related Mouse Cursor Motions by Conversion Type

	April	Dan	Christy	Lisa	Total
Decimal to Fraction	5	0	7	0	12
Fraction to Decimal	3	1	5	0	9
Total	8	1	12	0	21

Table 28

Frequency Counts of Mouse Cursor Motions Indicating Alignment for each Type of Applet

	Dual Construction	One-way Labeled	One-way Unlabeled	Total Gestures
Decimal to Fraction	4	8	0	12
Fraction to Decimal	3	4	2	9
Total	7	12	2	21

Table 28 above displays the frequency count of the four students' *mouse cursor motions indicating alignment* while using each of the three types of conversion applets for conversions of decimals to fractions and fractions to decimals. In a pattern similar to that for hand gestures indicating alignment, the largest number of mouse cursor motions indicating alignment occurred while the students used the one-way labeled applets and the lowest number of these mouse cursor motions occurred while the students were using the one-way unlabeled applets.

The researcher coded five mouse cursor motions as indicating alignment as participants used the *comparison* applet to compare fractions and decimals, all of which were attributed to Christy. As was the case with hand gestures, a possible explanation of Christy's *mouse cursor motions indicating alignment* while using the *comparison* applet is that she produced this gesture as part of her reasoning for why one quantity is greater than another on the two number lines, and thus such points do not line up.

Explanations when students mentioned alignment. When reviewing transcripts of students' verbal explanations during clinical interview sessions, the researcher coded *mention of alignment* for any explanation where students mentioned the alignment (or lack of alignment) of points on the two number lines of the applets. The following is a transcript excerpt from the third task-based interview with April, in which she was presented with the task of converting 0.04 to a fraction in the context of a *dual construction* applet:

086	Interviewer	So, this time it gives you, so yeah, given that one one hundredth is point Oh one, find the fraction for point Oh four.
087	April	I just move that one here, and out of twenty-fifth.
088	Interviewer	Yeah, twenty-fifths.
089	April	Twenty-five would be right there [April uses the applet to make 0.04 and 4/25].
090	April	Um, that's not right, they don't line up, which means I probably did the math wrong.

We can see that in line 090 of the above excerpt that after April used the applet to make both 0.04 and 4/25 that she noticed the points and lengths did not line up. April's explanation here was coded as *mentioning alignment* because of her observation. Thus,

Table 29

Frequency Counts of Participants' Mentions of Alignment while Using the Applets

	Dual Construction	One-Way Labeled	One-Way Unlabeled	Total
April	7	4	1	12
Dan	1	1	0	2
Christy	2	8	0	10
Lisa	0	0	0	0
Total	10	13	1	24

Table 30

Frequency Counts of Participants' Mentions of Alignment by Conversion Type

	April	Dan	Christy	Lisa	Total
Decimal to Fraction	10	1	7	0	18
Fraction to Decimal	2	1	3	0	6
Total	12	2	10	0	24

April was able to use visual feedback from the applet in the form of lack of alignment of the points and corresponding lengths to determine that the fraction 4/25 she made using the sliders was not equivalent to the given decimal 0.04.

Table 29 shows the frequency count of mentions of alignment as the participants used each of the three types of applets. As was the case with alignment-related mouse cursor motions, all of the alignment-related mentions during explanations were attributed to April, Dan, and Christy, whereas Lisa did not mention alignment while using the applets.

Table 30 shows the frequency count of participants' mentions of alignment during both conversions of decimals to fractions and fractions to decimals. Table 30 indicates that students mentioned alignment more frequently during conversions of decimals to fractions than during conversions of fractions to decimals.

Table 31 below shows the frequency count of students' mention of *alignment* while using the different conversion applets as well as the frequency count of students'

Table 31

Frequency Count of Students' Mentions of Alignment during Conversion Tasks

	Dual Construction	One-way Labeled	One-way Unlabeled	Total Mentions
Decimal to Fraction	6	11	1	18
Fraction to Decimal	4	2	0	8
Total	10	13	1	26

mentioning of *alignment* during conversions of decimals to fractions and fractions to decimals.

We see from Table 31 that students mentioned alignment of points during 18 explanations of decimals to fractions whereas students mentioned alignment during 6 conversions of fractions to decimals. As in the case of hand gestures indicating alignment, a possible explanation of this finding is that students may have drawn on the *alignment* affordance of the applets more often during conversions of decimals to fractions than during conversions of fractions to decimals. Table 25 indicates that students mentioned alignment many more times when using the dual construction and one-way labeled applets than when using the one-way unlabeled applets. This pattern is similar to that for hand gestures and mouse cursor motions indicating alignment, where students produced fewer of these types of responses while using the one-way unlabeled applets.

April was the only student who mentioned *alignment* twice while using the comparison applet. This finding is again consistent with the alignment-related hand gestures of Christy and Lisa and mouse cursor motions of Christy as these participants used the comparison applet. This indicates that students made sense of comparison tasks in terms of alignment of points on number lines as they used the comparison applet.

Partition Affordance

Hand gestures indicating partition. The researcher coded a hand gesture as a *partition* gesture if the gesture appeared to indicate points on the number line, such as a horizontal hopping motion with a hand or forefinger. Students made many gestures consistent with *partition* while using the applets for conversions, and Table 32 shows the frequency count of the number of partition-related hand gestures produced by the four participants as they used the three types of applets. Table 32 indicates that each of the four students produced partition-related hand gestures as they used the applets.
Table 33 shows the frequency-count for the partition-related hand gestures of each of the four participants by the type of conversion, decimal to fraction versus fraction

Table 32
Participant Frequency Counts of Partition-related Hand Gestures by Applet Type

	Dual Construction	One-Way Labeled	One-Way Unlabeled	Total
April	4	2	4	10
Dan	1	1	0	2
Christy	0	4	6	10
Lisa	0	0	5	5
Total	5	7	15	27

Table 33

Participant Frequency Counts of Partition-related Hand Gestures by Conversion Type

	April	Dan	Christy	Lisa	Total
Decimal to Fraction	5	1	8	4	18
Fraction to Decimal	5	1	2	1	9
Total	10	2	10	5	27

to decimal. Table 33 indicates that the students produced more partition-related hand gestures during decimal to fraction conversions than for fraction to decimal conversions.

Table 34 shows the frequency counts of students' *partition* gestures for each of the three types of conversion applets as well as the frequency count for each type of applet for conversions of fractions to decimals and decimals to fractions.

In a pattern similar to that of the findings regarding *alignment* gestures, we can see in Table 34 that students produced more gestures indicating the *partition* feature of the applets during decimal to fraction conversions than during fraction to decimal conversions. Specifically, when using each of the three applet types, students produced 18 partition-related gestures during decimal to fraction conversions and 9 partition-related gestures during fraction to decimal conversion. This finding would appear to indicate that students drew on the *partition* feature of the applets more often during decimal to fraction conversions than during fraction to decimal conversions, which appears to indicate the increased use of the partition features of the applet for making

Table 34

Frequency Counts of Partition-Related Hand Gestures for each Type of Applet and Conversion type

	Dual Construction	One-way Labeled	One-way Unlabeled	Total Hand Gestures
Decimal to Fraction	4	5	9	18
Fraction to Decimal	1	2	6	9
Total	5	7	15	27

sense of decimal to fraction conversions.

Additionally, Table 34 indicates the prevalence of hand gestures indicating partition while the students used the one-way unlabeled applets. This finding is in contrast to findings regarding the alignment affordance, where students produced fewer hand gestures indicating alignment while using the one-way unlabeled applets than when using the dual construction and one-way labeled applets.

Mouse cursor motions indicating partition. The researcher coded a student's movement of the mouse as a *mouse cursor motion indicating partition* if the student used the mouse to move the cursor over partition points of the number lines or hovered the mouse over a series of partition points of the number lines. For example, if a student appeared to be using the mouse cursor to count partition points of a number line this was coded as a partition mouse cursor motion. Table 35 shows the frequency count of participants' partition-related mouse cursor motions as they used the three types of applets. Table 35 indicates that each of the four participants produced partition-related mouse cursor motions while using the applets.

Table 35

Participant Frequency Counts of Partition-related Mouse Cursor Motions by Applet Type

	Dual Construction	One-Way Labeled	One-Way Unlabeled	Total
April	0	0	13	13
Dan	1	2	9	12
Christy	3	6	12	21
Lisa	4	0	2	6
Total	8	8	36	52

Table 36 shows the frequency counts of participants' partition-related mouse cursor motions for both of decimal to fraction conversions and fraction to decimal conversions. Table 36 indicates that participants produced somewhat more partition-related mouse cursor motions for decimal to fraction conversions than for fraction to decimal conversions.

Table 37 shows the frequency count of students' *mouse cursor motion indicating partition* for each of the three types of conversion applets, including the conversions of fractions to decimals and decimals to fractions.

Table 36

Participant Frequency Counts of Partition-related Mouse Cursor Motions by Conversion Type

	April	Dan	Christy	Lisa	Total
Decimal to Fraction	5	5	8	3	21
Fraction to Decimal	8	7	13	3	31
Total	13	12	21	6	52

Table 37

Frequency Counts of Partition Mouse Cursor Motions for each Applet Type and Type of Conversion

	Dual Construction	One-way Labeled	One-way Unlabeled	Total Gestures
Decimal to Fraction	2	6	13	21
Fraction to Decimal	6	2	23	31
Total	8	8	36	52

Table 37 indicates the prevalence of *mouse cursor motions indicating partition* during students' use of the *one-way unlabeled* applets. A likely explanation for this finding is that students' attended to the partition points of the unlabeled number line in their attempts to determine the quantity represented on the number line as an unlabeled point and length. Thus, students attended to the partition feature of the applets more when using the *one-way unlabeled* applets than when using either the *one-way labeled* applets or the *dual construction* applets. This finding is in contrast with the findings regarding students' mouse cursor motions indicating alignment as they used the one-way unlabeled applets. In particular, students produced more mouse cursor motions indicating alignment while using the dual construction and one-way labeled applets and fewer mouse cursor motions indicating alignment while using the one-way unlabeled applets.

Explanations when students mentioned partition. Students mentioned the *partition* affordance of the number lines when using the conversion applets. When coding the transcript data, the researcher coded an explanation as *mentioning partition* when

Table 38

Frequency Counts of Participants' Mentions of Partition while Using the Applets

	Dual Construction	One-Way Labeled	One-Way Unlabeled	Total
April	2	4	8	14
Dan	0	5	0	5
Christy	0	2	5	7
Lisa	0	0	0	0
Total	2	11	13	26

their explanations referred to partition points of either of the parallel number lines. The following is a transcript excerpt from the third task-based interview with Christy, in which she was working with a *one-way unlabeled* applet and was presented with the task of converting 0.55 to the fraction 11/20:

> Interviewer: Okay, so how did you know that was point five-five? You got that one pretty quick.
>
> Christy: Well, because I counted the dots. So then I did one two three four five, and then there's one in the middle, so then I thought, "hmmm, that's probably point five plus point zero five." So then I plussed those together and then that's point five five.

The researcher coded Christy's explanation as *mentioning partition* for two reasons. The first reason is that in the first sentence she says she "counted the dots." The second reason is that she subsequently mentioned "there's one in the middle," which refers to a displayed point between the points on the decimal number line located at 0.5 and 0.6, which she used to deduce that the point and length must represent the quantity 0.55.

Table 38 shows the frequency counts of participants' mentioning of partition-related features of the applets as they used the three different types of applets. Table 38

Table 39

Frequency Counts of Participants' Mentions of Partition by Conversion Type

	April	Dan	Christy	Lisa	Total
Decimal to Fraction	10	4	4	0	18
Fraction to Decimal	4	1	3	0	8
Total	14	5	7	0	26

shows that April, Dan, and Christy all mentioned partition-related features of the applets as they used them, whereas Lisa did not mention partition-related features while using the applets.

Table 39 displays the frequency counts of students' mentioning of partition-related features of the applets for both conversions of decimals to fractions and conversions of fractions to decimals. Table 39 indicates that each of April, Dan, and Christy mentioned partition-related features of the applets more often during conversions of decimals to fractions than during conversions of fractions to decimals.

Table 40

Frequency Counts of Students' Mentions of Partition Using the Three Types of Conversion Applets

	Dual Construction	One-Way Labeled	One-Way Unlabeled	Total Gestures
Decimal to Fraction	2	10	6	18
Fraction to Decimal	0	1	7	8
Total	2	11	13	26

Table 40 shows the frequency count of students' *mentions of partition* while using each of the three types of conversion applets as well as the direction of the conversion (decimal to fraction or fraction to decimal).

From Table 40 we can see that students referred to the *partition* feature of the applets 18 times during decimal to fraction conversions whereas they mentioned the partition features 8 times during conversions of fractions to decimals. This finding again lends support to the conclusion that students drew on the *partition* affordance of the applets more often during decimal to fraction conversions than during fraction to decimal conversions.

CHAPTER V

DISCUSSION

This chapter is comprised of six sections. The first section discusses the findings concerning students' reasoning regarding the relationship between fractions and decimals. The second section discusses the findings concerning the affordances of the applets that supported students' reasoning regarding the decimal-fraction relationship. The third section describes the contributions of this study to what is known about how students reason about the relationship between fractions and decimals. The fourth section provides implications of this study for educators. The fifth section discusses limitations of the study. The sixth section provides recommendations for future research into students' reasoning regarding the decimal-fraction relationship.

Conceptions of the Decimal-Fraction Relationship

Previous research has observed that some students hold the synthetic model that fractions and decimals are different types of numbers and that there is no relationship between fractions and decimals (Markovits & Sowder, 1991; Vamvakoussi & Vosniadou, 2010). In contrast, each of the students in this study believed that fractions could be expressed as decimals and decimals could be expressed as fractions. The students' conceptions of the relationship between fractions and decimals is primarily a collection of conversion procedures, in the sense that the students conceived of the decimal-fraction relationship in terms of procedures for converting between fractions and decimals. For instance, in describing why a fraction and a decimal are equivalent, students did not use language to describe how both represent the same quantity based on the underlying concept of fraction equivalence. Moreover, during tasks of comparing fractions and decimals, the students commonly listed reasons why one quantity was greater than another in terms of conversions of quantities.

Two of the students, April and Christy, were observed having synthetic models regarding the relationship between fractions and decimals, specifically concerning the relationship between the fraction 1/8 and the decimal equivalent 0.125. The specific synthetic model of April and Christy was that it is not mathematically accurate or correct to express a fraction in the form 12.5/100, where the numerator of the fraction consists of a decimal. Both April and Christy displayed reluctance to accept the idea that 12.5/100 is a mathematically legitimate expression of a rational number.

Several conclusions can be drawn concerning the students' use of strategies for converting between fractions and decimals. One observation is that students used a wide variety of strategies to convert between fractions and decimals and were able to flexibly choose conversion strategies depending on the type of conversion. Indeed, students used 11 different strategies in their explanations of conversions between fractions and decimals. The documented variety of strategies is consistent with the finding of Smith (1995) that competent reasoning with rational numbers "depends on a much richer and more diverse knowledge base that includes many numerically specific and invented strategies, as well as general strategies learned from instruction. These strategies are richly connected and flexibly applied to solve problems" (p. 3).

Those conversion strategies that made use of multiplication number facts are related to *number specific computational resources,* an idea described by Sherin and Fuson (2005). In the context of strategies for multiplication, Sherin and Fuson consider *number specific computational resources* to be students' in depth knowledge about multiplication for specific numbers that can become the basis for their use of new multiplication strategies, and where learned multiplication number facts are examples of number specific computational resources. Sherin and Fuson maintain that changes in students' multiplication strategies are often driven by changes in their number specific computational resources.

Students were clearly employing number specific computational resources during those strategies in which they drew on their knowledge of number facts during conversions, including the strategies of *scaling up*, *reducing*, *multiplication of units*, *addition of units from a base unit*, and *scaling up* and *adding or subtracting units*. Indeed, Sherin and Fuson identify a variety of multiplication strategies that they refer to as *hybrid strategies*, which are relevant to the present study. Indeed, *learned product + additive calculation* is a hybrid strategy identified by Sherin and Fuson that is closely related to the *scaling up and adding or subtracting units* conversion strategy of the present study. Sherin and Fuson describe *learned product + additive calculation* as a strategy students use to find products in which they use a multiplication number fact to get partway to the result and then an additive calculation to find the final product. As an example of the *learned product + additive calculation* strategy, a student might find the product 7 x 8 by using a known number fact of 7 x 7 = 49, and then adding 7 more to obtain 56 as the product.

In the current mathematics curriculum, students begin learning about the important concepts of ratio and proportion during the sixth grade (Common Core State Standards Initiative, 2010). Consequently, it makes sense to consider the relationship between the results of the present study and the findings from research of student-invented proportional reasoning strategies. In fact, finding the decimal equivalent of a fraction or the fraction equivalent of a decimal can be considered as solving a proportion. For example, to find the decimal equivalent of 3/4 is mathematically equivalent to finding the unknown numerator x in the proportion $\frac{3}{4} = \frac{x}{100}$. In fact, the various student-invented *build-up processes* for solving proportion problems described by Kaput and West (1994) are strategies that bear resemblance to some of the conversion strategies described in the present study. The basic *build-up process* that was used by students in the study of Kaput and West to solve proportions is a kind of coordinated double skip counting of quantities that bears some resemblance to the *addition of units from a base unit* conversion strategy identified in the present study. Moreover, the *abbreviated build up process* observed by Kaput and West is a strategy in which students use multiplication of quantities to solve proportions is similar to the *multiplication of units* strategy observed in the present study.

Many of the conceptual stepping-stones for understanding the decimal-fraction relationship described in Chapter 2 are reflected in the students' strategies for converting between fractions and decimals. Students were observed directly using strategies based on three of the conceptual stepping-stones, namely *halving*, *doubling*, and *disembedding*.

Students also leveraged their benchmark knowledge to recognize many fraction-decimal equivalences, and used the *benchmark and unit strategy* to find equivalences for non-benchmark quantities. Regarding the conceptual stepping stone of unit fraction and decimal magnitudes, students demonstrated their understanding of this stepping-stone as they made fraction-decimal conversions using strategies based on unit fractions and their decimal equivalents, which included *multiplication of units*, *addition of units from a base unit*, *scaling up and adding or subtracting units*, *halving*, *doubling*, and *disembedding*. It was evident that students employed the conceptual stepping-stone of equivalence and the decimal fraction relationship by using conversion strategies based on fraction equivalence, which included *scaling up* and *reducing*. Additionally, the conversion strategies of *multiplication of units* and *addition of units from a base unit* are related to the conceptual stepping-stone of iteration.

Additionally, the findings indicate that students used conversion strategies in unexpected ways. *Scaling up* was not a strategy the researcher expected to observe students using during explanations of the conversion of decimals to fractions, for the following reason: Consider converting a decimal such as 0.36 to the fully reduced fraction 9/25. Such a conversion involves reducing the common factors from the numerator and denominator of 36/100, the fraction equivalent to 0.36. Carrying out this reduction involves finding the largest number (4) that is a common factor of both 36 and 100, and dividing each of these numbers by the factor. However, in a number of cases, participants apparently anticipated that the fraction would reduce to 9/25 and then reasoned that the *scaling factor* of four could be used to scale 9/25 up to 36/100, which yields 0.36. Students may have used such a strategy to avoid using division to reduce 36/100 to 9/25. Converting decimals to fractions by using *scaling up* appeared to allow the participants to draw on their knowledge of multiplication number facts. For instance, in the case of the conversion of 0.36 to 9/25, participants may have recalled the multiplication facts that allowed them to realize that 9 x 4 = 36 and that 25 x 4 = 100, and thus it must be true that 9/25 is equivalent to 36/100.

Another unexpected use of a conversion strategy was the use of the *reducing* strategy during conversions of fractions to decimals. Because the *reducing* strategy naturally lends itself to the conversion of decimals to fractions, it is expected for students to commonly use *reducing* during conversions of decimals to fractions, and to observe students using *reducing* less often during conversions of fractions to decimals. This is because decimals in tenths or hundredths become fractions over 10 or 100 (respectively) when expressed as fractions, and where unreduced fractions will have common factors in the numerator and denominator that can be reduced. However, the results indicate that students used the *reducing* strategy during several conversions of fractions to decimals. The students' unexpected use of conversion strategies further emphasizes the broad diversity of strategies the students used during conversion tasks and their flexibility in applying conversion strategies.

Furthermore, students were able to take advantage of an understanding of the relationship between unit fractions and their equivalent decimals as the basis for strategies of converting between fractions and decimals. Indeed, each of the five different strategies of *benchmark and unit, multiplication of units*, *addition of units from a base* *unit*, *scaling up and adding or subtracting units*, and *disembedding* are based on the relationship between a unit fraction and the decimal equivalent. The *Common Core State Standards for Mathematics* emphasize that students should understand fraction quantities as iterations of unit fractions, and thus students' use of the strategies involving the unit fraction-decimal relationship indicate that the conception of fractions as iterations of unit fractions can be used as a basis for understanding of the fraction-decimal relationship (Common Core State Standards Initiative, 2010).

The findings indicate that the students preferred to use conversion strategies based on their knowledge of multiplication and division number facts. This can be seen in the prominent use of the *scaling up* strategy for fraction to decimal conversions and the use of the *reducing* strategy for decimal to fraction conversions involving denominators of 5, 20, and 25. However, a different pattern was observed for conversions involving denominators of 8, where *reducing* was scarcely used and *scaling up* was not used at all. Indeed, because conversions involving denominators of 8 are much more complicated than denominators of 5, 20, and 25, students chose to rely primarily on either their *benchmark knowledge* or the *benchmark and unit* strategy, as these strategies were especially prominent during fraction to decimal conversions with denominators of 8.

Whether students were asked to convert a fraction to a decimal or a decimal to a fraction appeared to influence their choice of conversion strategy, where there are distinct patterns of strategy use for conversions of fractions to decimals and decimals to fractions. In particular, for conversions involving denominators of 5, 20, and 25, students used a greater variety of strategies for conversions of decimals to fractions than fractions to decimals. For denominators of 5, students used 4 types of strategies for fraction to decimal conversions, whereas they used 6 types of strategies for conversions of decimals to fractions. For denominators of 20, students used 5 types of strategies for conversions of fractions to decimals, whereas they used 9 types of strategies for conversions of decimals to fractions. For denominators of 25, students used 5 types of strategies for conversions of fractions to decimals, and used 6 types of strategies for conversions of decimals to fractions. However, this pattern reverses for conversions involving denominators of 8, since students used 7 types of strategies for conversions of fractions to decimals and 4 different strategies for conversions of decimals to fractions. A possible explanation for these differences in strategy use for conversions involving denominators of 5, 20, and 25 is that students may have had a better idea of which strategies to apply during fraction to decimal conversions and were more decisive in their approach to strategy choice than for conversions of decimals to fractions.

Affordances of the Number Line-Based Applets

The researcher investigated the affordances of the applets for supporting the students' reasoning regarding the decimal-fraction relationship by analyzing data from students' explanations, hand gestures, and mouse cursor motions that indicated students' attending to the features of the applets supporting their conversion reasoning. The affordances of *alignment* and *partition* emerged as the key affordances of the features of the applets that supported students' reasoning regarding the relationship between fractions and decimals.

In terms of the five categories of affordances of virtual manipulatives identified by Moyer-Packenham and Westenskow (2013), the applets afforded students' awareness of alignment, which belongs to the affordance category of efficient precision. The applets afford *alignment* by efficiently and precisely depicting equivalent fractions and decimals as points that align on parallel number lines. Furthermore, the applets afforded students' awareness of *partition* , which also belongs to the affordance category of efficient precision, since the applets efficiently and precisely depict and permit the manipulation of fraction and decimal quantities. Based on these two affordances, results of the study suggest that interactive applets incorporating parallel number lines can support students' reasoning regarding the relationship between fractions and decimals.

It is worth noting that the data indicates the students drew on the affordances of the applets more frequently during conversions of decimals to fractions than during conversions of fractions to decimals. For the *alignment* affordance, the students made more hand gestures indicating alignment, more mouse cursor motions indicating alignment, and more explanations referring to alignment during conversions of decimals to fractions than during conversions of fractions to decimals. For the *partition* affordance, the students made more hand gestures indicating partition and mentioned partition features more frequently during conversions of decimals to fractions than during conversions of fractions to decimals. Evidently, decimal to fraction conversions involve more student sense making than fraction to decimal conversions, where students were able to draw on the affordances of the applets in the process of such sense making, which would explain the differences in the observed frequencies of affordance-related gestures, mouse cursor motions, and statements for these two types of conversion tasks. Thus, because students increasingly drew on the affordances of the applets during decimal to fraction conversions, the researcher observes the potential of these applets for supporting students' reasoning particularly during conversions of decimals to fractions.

Contributions of the Study

This study makes three significant contributions to what is known about students' reasoning regarding the decimal-fraction relationship and the use of number line representations for supporting such reasoning.

First, this study demonstrates the potential suitability of using interactive applets incorporating parallel number lines for supporting students' reasoning about the relationship between fractions and decimals. Despite NCTM's (2000) recommendation in the *Principles and Standards for School Mathematics* for the instructional use of parallel number lines for teaching the rational number concepts of order and equivalence, the researcher is aware of no previous investigation of the use of parallel number lines for supporting students' reasoning regarding order and equivalence concepts for the decimal-fraction relationship.

Second, this study provides detailed empirical evidence of fifth-grade students' strategies for converting between fractions and decimals, demonstrating students' ability to flexibly apply a variety of conversion strategies depending on the nature of the given conversion task. Results of this study build on the results reported in Moss and Case (1999) and Smith (1995). Indeed, Moss and Case found that fourth-grade students were able to invent the strategies of halving and doubling to make conversions between fractions and decimals for specific fractions with terminating decimal representations. Findings of the current study are consistent with those of Smith (1995), who documented students' use of a wide variety of' strategies for tasks involving order and equivalence of fractions for students ranging in ages from elementary school through high school. However, the current study further revealed students' strategies of order and equivalence in the context of the decimal-fraction relationship. The findings reported in this study contribute to the research literature by indicating students' preference for using conversion strategies based on their knowledge of multiplication and division number facts. These findings highlight the fundamental role that students' procedural knowledge of multiplication and division number facts can play in supporting their understanding of the relationship between fractions and decimals. Additionally, this study contributes to the research literature on reasoning and knowledge of rational numbers by demonstrating that students use differing patterns of conversion strategies depending on whether they are converting decimals to fractions or fractions to decimals.

Third, this study contributes to the research literature by using a combination of students' explanations, hand gestures, and mouse cursor motions to provide evidence of the affordances of virtual manipulatives for supporting students' mathematical reasoning. This was carried out by coding data in instances where students' explanations, hand gestures, or mouse cursor motions indicated their attention to particular features of the applets while they were engaged in equivalence and order tasks involving pairs of fractions and decimals. This analysis yielded evidence of the affordances of *alignment* and *partition* for supporting students' reasoning regarding the relationship between fractions and decimals, which has not previously been documented in the research literature.

Implications for Instruction

Results of this study have several implications for instruction regarding the relationship between fractions and decimals for fractions with terminating decimal representations. First, students are capable of inventing and using a wide variety of strategies for converting between fractions and decimals, and that different students may prefer to use different conversions strategies. Teachers can encourage their students to use a variety of strategies for converting between fractions and decimals to facilitate the development of students' *rational number sense*.

Second, students have a tendency to use procedural conversion strategies based on their knowledge of multiplication and division number facts. Indeed, this finding suggests multiplication and division number facts can play an integral role in enhancing students facility at converting between fractions and decimals, and underscores the importance of these number facts in supporting students' understanding of the decimal-fraction relationship. Thus, teachers should consider initiating instruction of conversions between fractions and decimals only after the students have a thorough understanding of multiplication and division number facts.

Third, conversions from decimals to fractions are very different than conversions of fractions to decimals for some students. Specifically, because students may have a better understanding of conversions of fractions to decimals than conversions of decimals to fractions, they used fewer and potentially more familiar strategies for conversions of fractions to decimals than for decimals to fractions. Thus, to ensure that students are equally versed in both types of conversions, teachers can devote equal emphasis to facilitating students' understanding of each of the two types of conversions, fractions to decimals and decimals to fractions.

Fourth, conversions between fractions and decimals that involve denominators of 8 are much more challenging for students than conversions involving denominators of 5, 20, and 25. This is because for conversions of fractions with denominators of 8, students are less able to draw on their knowledge of multiplication and division number facts to make conversions between fractions and decimals. However, the researcher suggests that conversions between fractions and decimals involving fractions with denominators of 8 can form the basis for challenging activities for students. Such activities could be designed to enrich students' understanding of the relationship between fractions and decimals by fostering understanding in ways apart from conversion strategies based on knowledge of multiplication and division number facts.

The fifth suggestion is that students' understanding of the relationship between a unit fraction and its decimal equivalent can inform instructional approaches for developing an overall understanding of the decimal-fraction relationship. Specifically, based on an understanding of a unit fraction and its decimal equivalent, students are capable of inventing numerous strategies to find other fraction-decimal equivalences.

Students were able to apply the *disembedding* strategy during some conversions when asked to provide an additional conversion strategy, even though this strategy did not appear to readily occur to the students. Thus, the sixth suggestion is that the *disembedding* strategy may be a productive strategy for students to learn for converting between fractions and decimals.

Seventh, virtual manipulatives incorporating parallel number lines can form the basis for tasks and activities for converting between fractions and decimals that are capable of supporting students' reasoning about the relationship between fractions and decimals. Indeed, the researcher was able to easily use GeoGebra to develop the applets used in this study, which suggests the possibility that elementary and middle school teachers could develop similar applets for their students' use. Another possibility is the wider dissemination to school teachers of similar applets developed using GeoGebra for use during rational number instruction.

Limitations

There are some limitations of this study, most of which pertain to its nature as an exploratory study. One limitation concerns the small sample size, where the researcher was able to gather complete data sets from only four students. There are also limitations of the study regarding the nature of the sample of students. The sample of students came from a charter school affiliated with a research university, and thus the students are not representative of the overall population of fifth-grade students. Thus, results of the study would likely have differed if the researcher had selected students from a different school.

Furthermore, the students received excellent mathematics instruction from their classroom instructor, which resulted in their understanding of several aspects of the relationship between fractions and decimals. Results of this study might have differed if the study had taken place earlier in the school year, since the students would likely have known less about the decimal-fraction relationship. Lastly, due to the constraints in resources, the researcher was the single person who coded and analyzed the data. Thus, there was limited control for the likelihood of the researcher's biases, and because of these limitations, the findings of this study cannot be generalized.

Suggestions for Further Research

This study was exploratory in nature, and replicating the study to overcome the limitations is warranted to confirm the findings. Results of a similar investigation with students with less knowledge of the relationship between fractions and decimals would provide additional insights into students' reasoning about the decimal-fraction relationship. Furthermore, the purpose of a similar study could be to measure and study learning gains regarding the relationship between fractions and decimals and the role that number line-based applets can play in their learning of this relationship. Similar studies could also investigate the learning progression of students' conversion strategies as well as the genesis of the strategies. In particular, for students displaying synthetic models regarding the decimal-fraction relationship, a subsequent study could investigate the potential of the number line base applets for remediating their synthetic models and misconceptions.

An issue that this study leaves unaddressed is whether there is a relationship between the students' conversion strategies and the affordances of the applets, which could be the subject of a subsequent study. One particular question that could be investigated is whether there are relationships between students' conversion strategies and either their alignment- or partition-related hand gestures, mouse cursor motions, or explanations.

Lastly, the researcher believes that conversions between fractions and decimals involving fractions with denominators of 8 provide valuable insights into students' conversion strategies because their strategies for these types of conversions made little use of multiplication and division number facts. For this reason, the researcher recommends additional studies to investigate students' reasoning and strategies for conversions between fractions and decimals for fractions with denominators of 8.

Conclusion

Prior to this study, little was known about students' conceptions of fractions as decimals and decimals as fractions for fractions with terminating decimal representations, and how virtual manipulatives incorporating parallel number lines can support students' reasoning regarding the relationship between fractions and decimals. Thus, the researcher's purpose for conducting this dissertation study was to investigate these gaps in the research literature.

Several conclusions can be drawn from this exploratory study. First, students are able to flexibly use many different types of strategies for converting between fractions and decimals. Second, students appeared to prefer to use computational conversion strategies based on multiplication and division number facts. Third, there were differing patterns of conversion strategies depending on whether students were converting fractions to decimals or decimals to fractions. Fourth, the type of denominator appeared to play a role in the types of strategies students used for conversions. In particular, conversions involving fractions with denominators of 8 were especially challenging for students and resulted in a different pattern of conversion strategies than for conversions involving fractions with denominators of 5, 20, and 25. Fifth, instructional strategies based on students' understanding of a unit fraction and its decimal equivalent have potential to form the basis of instruction of the decimal-fraction relationship more generally. Sixth, that alignment and partition emerged as key affordances of the number line-based applets for supporting the students' reasoning regarding the relationship between fractions and decimals for fractions with terminating decimal representations. Alignment was achieved by focusing and constraining students' attention on particular fraction and decimal equivalences as the alignment of points on the parallel number lines. Additionally, partition was achieved through the efficient and precise representation of fraction and decimal quantities on the parallel number lines, where students were able to efficiently and precisely interact with those fraction and decimal quantities using the number line-based applets.

Subsequent research with different groups of students can validate the results found in this exploratory study and further investigate the potential of number line-based applets for supporting students' learning and understanding of the relationship between fractions and decimals.

REFERENCES

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296–333). New York: Macmillan.

Behr, M., Lesh, R., Post, T., & Silver E. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes*, (pp. 91-125). New York: Academic Press.

Behr, M. J., Wachsmuth, I., & Post, T. R. (1985). Construct a sum: A measure of children's understanding of fraction size. *Journal for research in mathematics education*, *16*(2), 120-131.

Behr, M. J., Wachsmuth, I., Post, T., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education*, *15*(5), 323-41.

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, *37*(4), 247-253.

Burlamaqui, L., & Dong, A. (2014). The use and misuse of the concept of affordance. In J. S. Gero (Ed.), *Design Computing and Cognition DCC'14*. Springer.

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology, 105*(2), 380-400.

Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Jr., Lindquist, M. M., & Reys, R. E. (1981). *Results from the second mathematics assessment of the National Assessment of Educational Progress*. Reston, VA: National Council of Teachers of Mathematics.

Carpenter, T. P., Fennema, E., & Romberg, T. A. (1993). *Rational numbers: An integration of research*. Hillsdale, N.J: Lawrence Erlbaum Associates.

Chi, M. T. (1997). Quantifying qualitative analyses of verbal data: A practical guide. *The Journal of the Learning Sciences*, *6*(3), 271-315.

Chinn, C. A., & Sherin, B. L. (2014). Microgenetic methods. In K. Sawyer (Ed.), *Cambridge handbook of the learning sciences* (2nd ed., pp. 171-190). Cambridge: Cambridge University Press

Clements, D. H., & Sarama, J. (2007). Early childhood mathematics learning. In F.K.

Lester (Ed.), *Second handbook of research on mathematics teaching and learning: a project of the National Council of Teachers of Mathematics* (pp. 629-667). Charlotte, NC: Information Age Pub.

Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Cramer, K. A., Post, T. R., & delMas, R. C. (2002). Initial fraction learning by fourth-and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the rational number project curriculum. *Journal for Research in Mathematics Education*, *33*(2), 111-144.

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction*, *17*(3), 283-342.

Fosnot, C. T., & Dolk, M. (2002). *Young mathematicians at work, 3: Constructing fractions, decimals, and percents*. Portsmouth, NH: Heinemann.

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., Jordan, N. C., Siegler, R., Gersten, R., & Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology*. Advance online publication. doi: 10.1037/a0032446

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge: Cambridge University Press.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine Publishing Company

Goldin, G. (2002). Representation in mathematical learning and problem solving. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 197-218). Mahwah, N.J: Lawrence Erlbaum.

Goldin-Meadow, S. (2003). *Hearing gesture: How our hands help us think*. Cambridge, Mass: Belknap Press of Harvard University Press.

Greeno, J. G. (1983). Conceptual entities. In D. Gentner & A. L. Stevens (Eds.), *Mental models* (pp. 227-252). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Hiebert, J. (1992). Mathematical, cognitive, and instructional analyses of decimal fractions. In G. Leinhardt, R. T. Putnam, R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 283-322). Hillsdale, NJ, England: Lawrence Erlbaum Associates, Inc.

Hiebert, J., & Behr, M. (1988). Introduction: Capturing the major themes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 1–18). Reston, VA: National Council of Teachers of Mathematics and Lawrence Erlbaum Associates.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York: Macmillan.

Hiebert, J., & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 199–223). Hillsdale, NJ: Erlbaum.

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *The Journal of Mathematical Behavior*, *14*(4), 365-378.

Kaput, J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). Albany: SUNY Press.

Kerslake, D. (1986). *Fractions: Children's strategies and errors*. *A report of the strategies and errors in secondary mathematics projec*t. NFER-NELSON Publishing Company, Ltd., Darville House, 2 Oxford Road East, Windsor, Berkshire SL4 1DF, England.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.

Kloosterman, P. (2010). Mathematics skills of 17-year-olds in the United States: 1978 to 2004. *Journal for Research in Mathematics Education*, *41*(1), 20-51.

Lamon, S. J. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel, J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 89-120). Albany, NY US: State University of New York Press.

Lamon, S. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education*, *27*(2), 170-193.

Lamon, S. (2001). Presenting and representating from fractions to rational numbers. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics*, 2001 Yearbook (pp. 146-165). Reston, VA: National

Council of Teachers of Mathematics.

Lamon, S. J. (2007). Rational numbers and proportional reasoning. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: a project of the National Council of Teachers of Mathematics* (pp. 629-667). Charlotte, NC : Information Age Pub.

Lehrer, R. (2003). Developing understanding of measurement. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *Research companion to the principles and standards for school mathematics* (pp. 179–192). Reston, VA: National Council of Teachers of Mathematics.

Lesh, R., & Lehrer, R. (2000). Iterative refinement cycles for videotape analyses of conceptual change. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 665-708). Mahwah, NJ: Erlbaum.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33–40). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, *21*(1), 16-32.

Mack, N. K. (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. P. Carpenter, E. Fennema, T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 85-105). Hillsdale, NJ, England: Lawrence Erlbaum Associates, Inc.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, *26*(5), 422-441.

Mamede, E., Nunes, T., & Bryant, P. (2005). The equivalence and ordering of fractions in part-whole and quotient situations. In H. L. Chick, & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education: Vol. 3*. (pp. 281–288). Melbourne: PME.

Markovits, Z., & Sowder, J. T. (1991). Students' understanding of the relationship between fractions and decimals. *Focus on Learning Problems in Mathematics*, *13*(1), 3-11.

Markovits, Z., & Sowder, J. (1994). Developing number sense: An intervention study in grade 7. *Journal for Research in Mathematics Education*, *25*(1), 4-29.

McNeill, D. (2005). *Gesture and thought*. Chicago: University of Chicago Press.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122-47.

Moyer, P. S., Bolyard, J. J., & Spikell, M. A. (2002). What are virtual manipulatives? *Teaching Children Mathematics*, *8*(6), 372-377.

Moyer-Packenham, P. S., & Bolyard, J. J. (2016). Revisiting the definition of a virtual manipulative. In P. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives* (pp. 1-16). New York: Springer.

Moyer-Packenham, P. S., & Westenskow, A. (2013). Effects of virtual manipulatives on student achievement and mathematics learning. *International Journal of Virtual and Personal Learning Environments, 4*(3), 35-50.

Moyer-Packenham, P. S., & Westenskow, A. (2016). Revisiting the effects and affordances of virtual manipulatives for mathematics learning. In K. Terry, & A. Cheney (Eds.), *Utilizing Virtual and Personal Learning Environments for Optimal Learning* (pp. 186-215). Hershey, PA: IGI Global. doi:10.4018/978-1-4666-8847-6.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. US Department of Education.

Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27-52.

Norton, A. H., & McCloskey, A. V. (2008). Modeling students' mathematics using Steffe's fraction schemes. *Teaching Children Mathematics*, *15*(1), 48-54.

Olive, J., & Lobato, J. (2008). The learning of rational number concepts using technology. In M. K. Heid & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Research syntheses* (Vol. 1, pp. 1-55). Charlotte, NC: Information Age Publishing

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education*, *14*(5) 307-317.

Post, T., Cramer, K., Behr, M., Lesh, R., & Harel, G. (1993). Curriculum implications of research on the learning, teaching and assessing of rational number concepts. In T. P. Carpenter, E. Fennema, T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 327-362). Hillsdale, NJ England: Lawrence Erlbaum Associates, Inc.

Post, T., Wachsmuth, I., Lesh, R. A., & Behr, M. J. (1985). Order and equivalence of rational numbers: A cognitive analysis. *Journal for Research in Mathematics Education*, *16*(1), 18-36.

Reimer, K., & Moyer, P. S. (2005). Third-graders learn about fractions using virtual manipulatives: A classroom study. *Journal of Computers in Mathematics and Science Teaching*, *24*(1), 5-25.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*(2), 346-62.

Rutledge, Z., Kloosterman, P., & Kenney, P. A. (2009). Mathematics skills and NAEP results over a generation. *Mathematics Teacher*, *102*(6), 445-451.

Sherin, B., & Fuson, K. (2005). Multiplication strategies and the appropriation of computational resources. *Journal for Research in Mathematics Education*, *36*, 347-395.

Siegler, R. S. (1996). *Emergent minds: The process of change in children's thinking*. New York: Oxford University Press.

Siegler, R. S. (2006). Microgenetic analyses of learning. In D. Kuhn & R. Siegler (Eds.), *Handbook of child psychology* (pp. 464–510). Hoboken, NJ: John Wiley & Sons.

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., ... & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade*. IES Practice Guide. NCEE 2010-4039. What Works Clearinghouse.

Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences*, *17*(1), 13-19.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273-296.

Smith, C. L., Solomon, G. A., & Carey, S. (2005). Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. *Cognitive Psychology*, *51*(2), 101-140.

Smith III, J. P. (1995). Competent reasoning with rational numbers. *Cognition and Instruction*, *13*(1), 3-50.

Sowder, J. T. (1995). Instructing for rational number sense. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 15-30). Albany: State University of New York Press.

Sowder, J. T., Bezuk, N., & Sowder, L. K. (1993). Using principles from cognitive psychology to guide rational number instruction for prospective teachers. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 239–258). Hillsdale, NJ: Lawrence Erlbaum.

Sowder, J. T., & Markovits, Z. (1989). Effects of instruction on number magnitude. In *Proceedings of Eleventh Annual Meeting: North American Charter of the International Group for the Psychology of Mathematics Education* (pp. 105-110).

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*(5), 503-518.

Steffe, L. P. (2003). Fractional commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 5. *The Journal of Mathematical Behavior*, *22*(3), 237-295.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95–114). Reston, VA: National Council of Teachers of Mathematics.

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics*, *66*(3), 273-291.

Vamvakoussi, X., Christou, K. P., Mertens, L., & Van Dooren, W. (2011). What fills the gap between discrete and dense? Greek and Flemish students' understanding of density. *Learning and Instruction*, *21*(5), 676-685.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, *14*(5), 453-467.

Vamvakoussi, X., & Vosniadou, S. (2007). How many numbers are there in an interval? Presuppositions, synthetic models and the effect of the number line. In S. Vosniadou, A. Baltas, & X. Vamvakoussi (Eds.), *Reframing the conceptual change approach in learning and instruction* (pp. 267–283). Oxford, UK: Elsevier.

Vamvakoussi, X., & Vosniadou, S. (2010). How many “decimals” are there between two “fractions”? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, *28*(2), 181-209.

Vamvakoussi, X., Vosniadou, S., Van Dooren, W. (2013). The framework theory approach applied to mathematical learning. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (2nd ed., pp. 305-321). New York and London: Routledge.

Vosniadou, S. (1994). Capturing and modeling the process of conceptual change. *Learning & Instruction*, *4*(1), 45-69.

Vosniadou, S., Vamvakoussi, X., & Skopeliti, I. (2008). The framework theory approach to the problem of conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 3-34). Mahwah, NJ: Lawrence Erlbaum.

APPENDICES

Appendix A

Institutional Review Board (IRB) Certificate

Institutional Review Board

USU Assurance: FWA#00003308

UtahState University
OFFICE OF RESEARCH AND GRADUATE STUDIES

Expedite #7

Letter of Approval

FROM:

Melanie Domenech Rodriguez, IRB Chair

Nicole Vouvalis, IRB Administrator

Nicole Vouvalis

To:	Sarah Brasiel, Scott Smith
Date:	May 06, 2015
Protocol #:	6603
Title:	Children's Construction Of Knowledge Of The Decimals-Fractions Relationship When Using Number Line Based Virtual Manipulatives
Risk:	Minimal risk

Your proposal has been reviewed by the Institutional Review Board and is approved under expedite procedure #7 (based on the Department of Health and Human Services (DHHS) regulations for the protection of human research subjects, 45 CFR Part 46, as amended to include provisions of the Federal Policy for the Protection of Human Subjects, November 9, 1998):

> Research on individual or group characteristics or behavior (including, but not limited to, research on perception, cognition, motivation, identity, language, communication, cultural beliefs or practices, and social behavior) or research employing survey, interview, oral history, focus group, program evaluation, human factors evaluation, or quality assurance methodologies.This approval applies only to the proposal currently on file for the period of one year. If your study extends beyond this approval period, you must contact this office to request an annual review of this research. Any change affecting human subjects must be approved by the Board prior to implementation. Injuries or any unanticipated problems involving risk to subjects or to others must be reported immediately to the Chair of the Institutional Review Board.

This approval applies only to the proposal currently on file for the period of one year. If your study extends beyond this approval period, you must contact this office to request an annual review of this research. Any change affecting human subjects must be approved by the Board prior to implementation. Injuries or any unanticipated problems involving risk to subjects or to others must be reported immediately to the Chair of the Institutional Review Board.

Prior to involving human subjects, properly executed informed consent must be obtained from each subject or from an authorized representative, and documentation of informed consent must be kept on file for at least three years after the project ends. Each subject must be furnished with a copy of the informed consent document for their personal records.

Appendix B

Clinical Interview Used for the Study

Interview Protocol Used for Initial and Final Interviews for the Dissertation Study

The purpose of this interview is to assess students' knowledge of the following rational number concepts:

- Benchmark values of fractions and decimals
- Place value concepts for decimals
- Order concepts of fractions and decimals
- Equivalence concepts of fractions
- Relationship between fractions and decimals
- Number lines
    - Locating fractions and decimals on number lines
- To determine if students mentally represent fractions in terms of concrete representations, such as circle models.

To gain an understanding of the strategies students use to solve these tasks: do students' strategies yield information about their understanding of fractions and decimals?

Introduction

- Introduce myself to the student
- Ask the student their name and what grade they are in
- Briefly explain the purpose of the interview to the student, which is to determine what they know about fractions and decimals. Explain that the student will not be judged on the correctness or incorrectness of their answers
- Explain to the student that the interview will be recorded for research purposes only, and that they should not be bothered by the presence of the camera

- The student will be asked to “think out loud” as they are performing each task
- After completing tasks, students may be asked to further clarify their thinking with prompts such as “Can you tell me how you thought about this problem?”
- The student will be provided with blank paper and a pencil, in case they wish to make any drawings or computations. Such notes will be kept as part of the data from the interview

Place Value Task – Construct a Decimal

Materials: Large numerals printed on card stock, including a decimal point.

Task: prompt the student to make various decimals, including tenths, hundredths, and thousandths, such as:

[Note: when you construct these cards, make one or some that are 0. so that students can then put their digits after that. This seems like a feasible way of implementing this task]

The student will be given these prompts:

- “Can you make the decimal eight-tenths?”
- “Can you make the decimal sixty-three one-hundredths?”
- “Can you make the decimal four hundred and thirty five thousandths?”

Decimal ordering tasks

Materials: Various decimals printed in large print on card stock

The student will be asked to order the fractions and to explain which is the smallest, and which is the largest

Task: The student will be given ordering tasks for fractions such as:

(Prompt the student to explain which is the smallest, and which is the largest, after they have ordered the fractions)

- Order three tenths decimals, such as 0.5, 0.6, and 0.7
- Order three hundredths decimals, such as 0.37, 0.45, and 0.62
- Order three tenths and hundredths decimals, such as 0.4, 0.34, and 0.53
- Order three tenths and hundredths decimals, where two are equivalent, such as 0.6, 0.60, and, 0.55

Fractions ordering tasks

Materials: Various fractions printed in large print on card stock

Task: The student will be given ordering tasks for fractions:

(Again, the student must be asked to explain which is the smallest and which is the largest)

- Order three fractions involving benchmark fractions: ¼, ½, and ¾
- Order three fractions with a benchmark fraction and two other fractions: 1/2, 2/5, and 3/5
- Order three fractions with the same numerator: 2/3, 2/4, and 2/5
- Order two fractions with a common difference between numerator and denominator: 4/5 and 5/6
- Order three fractions with a common difference between numerator and denominator: 2/3, ¾, and 4/5

- Order three fractions where two of the fractions are actually equivalent fractions: ½, and 2/4. [*The purpose of the task is to see if the student is actually able to identify the equivalent fractions*]

Construct a Fraction tasks

Materials: numerals printed on card stock, and a large card printed to suggest a fraction with locations to put the numerals in the numerator and denominator

- Task 1 – make the smallest possible fraction, given the numerals 1, 2, 3, 4, 5, 7, and 8
- Task 2 – make the fraction that is as close as possible to 1, given the numerals 1, 2, 3, 4, 5, 7, and 8
- Task 3 – make the fraction that is as close as possible to 1/2, given the numerals 2, 3, 5, 6, 7, 8, and 9

Construct equivalent fractions tasks

Materials: numerals printed on card stock, and a large card printed to suggest a fraction with locations to put the numerals in the numerator and denominator

Tasks: The student will be asked

- Construct another fraction that is equal to ½ given 2, 3, 4, 5, 6, 7, 8, and 9
- Construct another fraction that is equal to ¼ given 2, 3, 4, 5, 6, 7, 8, and 9

Fraction and Decimal Ordering tasks

Materials: Various fractions and decimals printed in large print on card stock

Task: The student will be given ordering tasks for fractions:

- Order ½ and 0.3

- Order 1/3 and 0.3
- Order 0.6 and ¾

Construct a Fraction Equivalent to a Given Decimal

Materials: numerals printed on card stock, and a large card printed to suggest a fraction and locations to put the numerals in the numerator and denominator, as well as some decimals also printed on card stock

The student will be shown the decimals shown below

Provide the student with the digits 1, 2, 4, 5, 7, 8, 10

Tasks:

- Construct a fraction that is equal to the decimal 0.5
- Construct a fraction that is equal to the decimal 0.25
- Construct a fraction that is equal to the decimal 0.75
- Construct a fraction that is equal to the decimal 0.1
- Construct a fraction that is equal to the decimal 0.2

If the student does not believe that such as construction is possible, ask them to explain why they believe that

Placing a Fraction on a Number Line Task

Provide the student with a large number line marked only with 0, ½, and 1

Task: Prompt the student to use their finger to locate where these rational numbers on the number line

- 5/6
- 1/6

- 3/7
- 5/7
- 0.1
- 0.8
- 0.75
- 0.25

Tell the student that this is all of the questions that I have for them, and thank them for their participation.

Appendix C

Sample Task for Comparing a Fraction and a Decimal

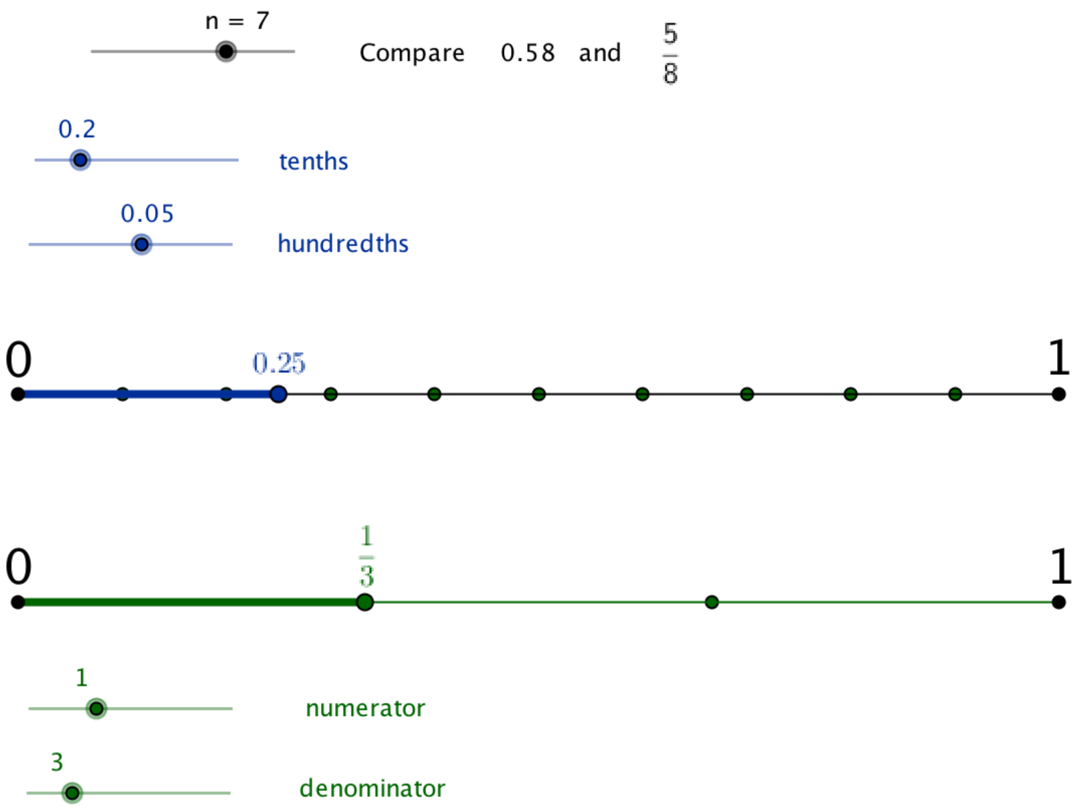

Figure 6. Screenshot of GeoGebra applet used in task in which students compare the value of pairs of fractions and decimals.

Figure 6 presents a screenshot of the GeoGebra applet used in tasks where students compare the values of pairs of fractions and decimals.

The uppermost slider (set at n = 7) is used to present new pairs of decimals and fractions for students to compare. The variable n for the slider ranges from 1 to 10, where changing the value of n on the slider presents a new pair of fractions and decimals for students to compare; hence, this applet allows students to compare the value of 10 pairs of fractions and decimals.

The next pair of sliders, shown in blue, control the tenths and hundredths values of the decimal, shown as a point, length (in blue), and decimal on the upper number line. For this task, students will need to use the blue tenths and hundredths sliders to construct the decimal 0.58.

The lowermost pair of green sliders below the two number lines control the value of a fraction, shown as a point, length (in green), and fraction on the lower number line.

Students succeed in this task by using the sliders to construct the decimal 0.58 on the upper number line and the fraction 5/8 on the lower number line, observing that because the length shown for the decimal 0.58 is shorter than the length for the fraction 5/8, which implies the decimal 0.58 is less than the fraction 5/8.

CURRICULUM VITAE

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EDUCATION

PhD May 2017
Instructional Technology and Learning Sciences, Utah State University
Concentration: Mathematics Education and Leadership
Specialization: Middle school children's learning of rational number concepts from virtual manipulatives

MS December 2010
Instructional Technology and Learning Sciences
Utah State University, Logan, Utah

MS June 1994
Mathematics
Emphasis: Differential Equations
University of Utah, Salt Lake City, Utah

BS June 1991
Mathematics
Emphasis: Pure Mathematics
Weber State University, Ogden, Utah

EMPLOYMENT HISTORY

WESTERN NEW MEXICO UNIVERSITY

Assistant Professor, Department of Mathematics and Computer Science (August 2015 – Present)

Responsibilities include teaching mathematics and mathematics education coursework, committee work, curriculum development, grant funded projects, collaboration with other faculty, and student advising.

UTAH STATE UNIVERSITY

Research Assistant and Lab Manager for the Active Learning Lab (January 2014 – Present)

Work as a research assistant for the Active Learning Lab, located in the Department of Instructional Technology and Learning Sciences. The principal investigators of the Active Learning Lab are Taylor Martin and Sarah Brasiel. Research assistant duties include work on the STEM Action Center Pilot Program, which is an evaluation of state-sponsored teacher professional development in the state of Utah. The Active Learning Lab has received funding from the state of Utah as a result of House Bill 0139, to carry out this evaluation. Primary job duties include: participating with the design and planning of studies and data analysis; collection of data through interviews, observation and testing; compilation and analysis of data; participation and preparation of reports and research writing. Mentoring junior research assistants concerning research methodologies and data analysis methods in the role of lab manager.

Research Assistant (July 2011 – May 2013)
Work as research assistant on the PAD Project, whose Principle Investigator is Victor Lee of the Department of Instructional Technology and Learning Sciences at Utah State University. The PAD Project is a design-based research project that is conducting an NSF funded multi-year project with fifth-grade students at Edith Bowen Laboratory School at Utah State University. The goals of the PAD Project are to create mathematics and science curriculum materials in which fifth-grade students explore designing experiments and gathering data from high-technology Physical Activity Devices (PAD) such as pedometers and heart rate monitors. The project has also investigated students' understanding of bodily motion by investigating the stop action animations students created using the SAM Animation software. Primary responsibility was as the archivist of collected data, which included encoding and archiving video recorded data and data from the SAM Animation experiments.

Adjunct Instructor of Mathematics at Utah State University (2003 – 2009)
Responsibilities included teaching undergraduate courses from the calculus sequence (Calculus I and II), algebra courses (Intermediate and College Algebra), Trigonometry, and a mathematics training course for pre-service elementary education teachers (Mathematics for Elementary Education Teachers).

Teaching Assistant for the Department of Mathematics-Statistics at Utah State University (1996 – 2003)
Responsibilities included teaching undergraduate courses from the calculus sequence (Calculus I and II), algebra courses (Intermediate and College Algebra), Trigonometry, and The Mathematics for Elementary Education Teachers course.

Adjunct Instructor of Mathematics at Weber State University (1996)
Responsibilities included teaching Beginning Algebra, Intermediate Algebra, and Contemporary Mathematics.

Adjunct Instructor of Mathematics at Salt Lake Community (1995 – 1996)

Responsibilities included teaching algebra courses (Intermediate and college algebra) and Advanced Engineering Mathematics.

AWARDS AND PROFESSIONAL RECOGNITION

- Recipient of Presidential Fellowship from the School of Graduate Studies at Utah State University for the 2011-2012 school year.
- Nominated for the Excellence in Instruction for First-Year Students Award by the Utah State University Retention Committee in 2006
- Robins Award Finalist for Graduate Student Teacher of the Year, 1999-2000, Utah State University
- Graduate Student Teacher of the Year, 1999-2000, Utah State University College of Science

RESEARCH

Research Interests:

- Children's learning and understanding of rational numbers
- Knowledge representation and mental models
- Use of instructional technologies such as virtual manipulatives for learning rational numbers
- Insights students' gestures provide concerning their conceptual and procedural knowledge of mathematics
- Investigation of conceptual change mechanisms as students learn rational number concepts
- Using self-explanations to improve students' conceptual understanding of rational numbers

Research Activities

- Conducted a pilot study of elementary and middle school children's understanding of rational number concepts of order and equivalence; in particular their understanding of the relationship between fractions and decimals, their understanding that fractions and decimals represent quantities, and their understanding of how to represent fractions and decimals on number lines
- Dissertation study, a microgenetic investigation involving clinical interviews of fifth-grade students. In the dissertation study, fifth-grade students are interviewed concerning their initial knowledge of fractions, decimals, and the fractions-decimals relationship. Subsequently, the students' participate in a tutorial sessions making use of constructivist tasks designed to improve their understanding of the relationship between fractions and decimals. During instructional sessions, students engage in tasks making use of number line based virtual manipulatives. After the instructional sessions, students are interviewed a second time to determine learning gains.

- Content analysis of fourth- and fifth-grade mathematics textbooks and curriculum materials. The purpose of this investigation is to document the textbooks' use of representations of fractions and decimals.

PUBLICATIONS

Conference Proceedings

Yuan, M., Kim, N. J., Drake, J., Smith, S., Lee, V. L. (2014, June). Examining how students make sense of slow-motion video. In J. L. Polman, E. A. Kyza, K. O'Neill, I. Tabak, W. R. Penuel, A. S. Jurow, K. O'Connor, T. Lee, & L. D'Amico (Eds.). *Proceedings of the 11th International Conference of the Learning Sciences* (ICLS) (pp. 1617-1618), Boulder, Colorado. ISSN# 1814-9316

Martin, T., Brasiel, S, Graham, D., Smith, S. B., Gurko, K., Fields, D., Smith. S. (2014, October). *FabLab professional development: Changes in teacher and student STEM content knowledge*. Paper presented at the Fablearn Conference, Stanford, CA.

Westenskow, A., Smith, S. (2014, November). *Using number lines in helping students develop fraction understanding*. Presentation at the Utah Council of Teachers of Mathematics, Layton, UT.

Smith, S., Lawanto, K., Brasiel, S., Martin, T. (2015, March). *Changing middle school teachers' algebra content knowledge and teaching self-efficacy beliefs through technology-enriched professional development*. Paper presented at the 26th International Conference of the Society for Information Technology and Teacher Education. March 2-6, 2015. Las Vegas, NV.

Brasiel, S., Martin, T., Smith, S.B., and Redington, S. (2015, April). *Developing Teacher Pedagogical Content Knowledge of Slope and Visual Representations for Solving Systems of Equations*. Paper presented at the 47th National Council of Supervisors of Mathematics Annual Conference. Boston, MA.

Smith, S., Brasiel, S., Martin, T. (2015, June). *Promoting teachers' integration of technology in teaching practice through professional development*. Paper presented at the 2015 International Conference for the International Society for Technology in Education. June 28 - July 1, 2015. Philadelphia, PA.

Graham, D., Brasiel, S., Martin, T., Smith, S. (2015, June). *FabLab professional development: Introducing a district to design-based learning*. Paper presented at the 2015 International Conference for the International Society for Technology in Education. June 28 - July 1, 2015. Philadelphia, PA.

UNIVERSITY TEACHING

Western New Mexico University (2015 – Present)
Department of Mathematics and Computer Science, College of Arts and Sciences

Courses Taught – Western New Mexico University
Mathematics 105 – Mathematics for the Liberal Arts
Undergraduate course. Students in majors outsides of the fields of mathematics, business, and the sciences learn concepts of mathematics applied to topics that include voting, power, apportioning, routing problems, and touring problems.

Mathematics 107 – Mathematics for School Teachers.
Undergraduate course. Education majors in fields outside of the sciences, mathematics, and business learn mathematics necessary to pass entrance exams for admission to the School of Education.

Mathematics 131 – College Algebra.
Undergraduate course. Students learning essential concepts of algebra including linear functions, quadratic functions, exponential functions, logarithmic functions, and transformations of functions.

Mathematics 171 – Calculus I.
Undergraduate course. Students learn fundamental calculus concepts, including limits, derivatives, rules for derivatives, applications of derivatives, antiderivatives, and definite and indefinite integrals.

Mathematics 172 – Calculus II.
Undergraduate course. Students learn calculus concepts, including methods of integration, applications of integration, infinite sequences and series, power series, differential equations, and calculus-based mathematical modeling.

Mathematics 301 – Understanding Elementary Mathematics I.
Undergraduate course. Pre-service teachers in early childhood education and Elementary Education learn mathematics content and methods for teaching basic mathematical concepts and procedures, including whole numbers, multidigit operations, fundamentals of number theory, integers, fractions, decimals, ratios, proportions, and percentages.

Mathematics 302 – Understanding Elementary Mathematics II.
Undergraduate course. Pre-service teachers in early ehildhood education and Elementary Education learn mathematics content and methods for teaching basic mathematical concepts and procedures, including linear equations, functions, probability, elementary data analysis, representations of data, fundamentals of geometry, and measurement.

Mathematics 304 – Mathematics for the Secondary Teacher.

Undergraduate course. Students learn a variety of approaches to the teaching of mathematics for preparation to teach secondary school mathematics.

Mathematics 327 – Survey of Geometry.
Undergraduate course. Students learn Euclidean geometry, geometry of dimensions, and non-Euclidean geometry.

Utah State University, Logan, Utah (1996 – 2009)
Department of Mathematics-Statistics, College of Science

Courses Taught – Utah State University
Mathematics 2020 – Mathematics for Elementary Education Teachers
Undergraduate course. Undergraduate students studying to become elementary education teachers learn strong mathematics fundamentals such as whole number operations, fractions, percentages, decimals, ratios, proportions, and geometry.

Mathematics 1220 – Calculus II
Undergraduate course. Students continue learning calculus topics such as methods of integration, fundamentals and applications of differential equations, infinite series and sequences, and vectors.

Mathematics 1210 – Calculus I
Undergraduate course. Students learn the basics of calculus, such as finding derivatives from the limit definition, properties of derivatives, applications of differentiation, and beginning methods of integration.

Mathematics 1060 – Trigonometry
Undergraduate course. Students learn the fundamentals of trigonometry and trigonometric functions, manipulating trigonometric identities, and applications of trigonometric functions.

Mathematics 1050 – College Algebra
Undergraduate course. Students continue learning fundamental algebra topics, such as transformations of the graphs of functions, quadratic functions, polynomials, rational functions, and exponential and logarithmic functions.

Mathematics 1010 – Intermediate Algebra
Undergraduate course. Students learn fundamental algebra topics, such as graphs of lines and linear functions, properties of exponents, radicals, inequalities, graphs of circles and ellipses, and rational functions.

PROFESSIONAL AFFILIATIONS

Member of the National Council of Teachers of Mathematics (NCTM)

Member of the American Mathematical Society (AMS)

Member of the International Society of the Learning Sciences (ISLS)

Member of the American Educational Research Association (AERA)